Semidefinite Programming Duality and Linear Time-invariant Systems

Venkataramanan (Ragu) Balakrishnan
School of ECE, Purdue University

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Workshop on Linear Matrix Inequalities in Control
LAAS-CNRS, Toulouse, France

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Joint work with Lieven Vandenberghe, UCLA

Basic ideas

- Many control constraints yield LMIs, many control problems are SDPs

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- From duality theory in convex optimization:
  - ⋆ Theorem of alternatives for LMIs
  - ⋆ SDP duality

Basic ideas

- Many control constraints yield LMIs, many control problems are SDPs

- LMIs are convex constraints, SDPs are convex optimization problems

- From duality theory in convex optimization:
  - ⋆ Theorem of alternatives for LMIs
  - ⋆ SDP duality

- Explore implication of convex duality theory on underlying control problem:
  - ⋆ New (often simpler) proofs for classical results
  - ⋆ Some new results

LMIs and Semidefinite Programming

- $\mathcal{V}$ is a finite-dimensional Hilbert space, $\mathcal{S}$ is a subspace of Hermitian matrices, $\mathcal{F} : \mathcal{V} \to \mathcal{S}$ is a linear mapping, $F_0 \in \mathcal{S}$
- Inequality $\mathcal{F}(x) + F_0 \geq 0$ is an LMI
- SDP is an optimization of the form:

$$\begin{array}{ll} \texttt{minimize:} & \langle c, x \rangle_{\mathcal{V}} \\ \texttt{subject to:} & \mathcal{F}(x) + F_0 \geq 0 \end{array}$$

A theorem of alternatives for LMIs

Exactly one of the following statements is true

1. $\mathcal{F}(x) + F_0 > 0$ is feasible

2. There exists $Z \in \mathcal{S}$ s.t. $Z \gneqq 0$, $\mathcal{F}^{\mathrm{adj}}(Z) = 0$, $\langle F_0, Z \rangle_{\mathcal{S}} \leq 0$

($\mathcal{F}^{\mathrm{adj}}(\cdot)$ denotes adjoint map, i.e., $\forall\, x \in \mathcal{V}$, $Z \in \mathcal{S}$, $\langle \mathcal{F}(x), Z \rangle_{\mathcal{S}} = \langle x, \mathcal{F}^{\mathrm{adj}}(Z) \rangle_{\mathcal{V}}$)

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- Variants available for nonstrict inequalities such as $\mathcal{F}(x) + F_0 \gneqq 0$ and $\mathcal{F}(x) + F_0 \geq 0$, and with additional linear equality constraints $\mathcal{F}(x) = 0$
- Typically get *weak* alternatives, need additional conditions (constraint qualifications) to make them strong

Proof of theorem of alternatives

LMI $\mathcal{F}(x) + F_0 > 0$ infeasible iff

$$F_0 \quad \notin \quad \mathcal{C} \stackrel{\Delta}{=} \{C \mid \mathcal{F}(x) + C > 0 \text{ for some } x \in \mathcal{V}\}$$

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$\mathcal{C}$ is open, nonempty and convex, so there exists hyperplane strictly separting F_0 and $\mathcal{C}$:

$$\exists Z \neq 0 \quad \text{s.t.} \quad \langle F_0, Z\rangle_{\mathcal{S}} < \langle C, Z\rangle_{\mathcal{S}} \quad \text{for all } C \in \mathcal{C}$$

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$$\exists Z \neq 0 \quad \text{s.t.} \quad \langle F_0, Z \rangle_{\mathcal{S}} < \langle x, -\mathcal{F}^{\mathrm{adj}}(Z) \rangle_{\mathcal{V}} + \langle X, Z \rangle_{\mathcal{S}} \quad \text{for all } x \in \mathcal{V}, X > 0$$

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Thus, there exists $Z \in \mathcal{S}$ s.t. $Z \gneqq 0$, $\mathcal{F}^{\mathrm{adj}}(Z) = 0$, $\langle F_0, Z\rangle_{\mathcal{S}} \leq 0$

Application: A Lyapunov inequality

- LMI $A^*P + PA < 0$ is feasible, or
- There exists Z s.t. $Z \gneqq 0, \quad AZ + ZA^* = 0$

Factoring $Z = UU^*$, can show

$$AU = US, \quad S \text{ has pure imaginary eigenvalues}$$

Thus:

LMI $A^*P + PA < 0$ is infeasible if and only if A has a pure imaginary eigenvalue

Other results

- "$P > 0,\ A^*P + PA < 0$" is infeasible iff $\lambda_i(A) \geq 0$ for some i
- "$A^*P + PA \lneqq 0$" is infeasible iff A is similar to a purely imaginary diagonal matrix
- "$A^*P + PA \leq 0,\ P \gneqq 0$" is infeasible iff $\lambda_i(A) \geq 0$ for all i
- "$A^*P + PA \lneqq 0,\ PB = 0$" is infeasible iff all uncontrollable modes of (A, B) are nondefective and correspond to imaginary eigenvalues
- "$P \gneqq 0,\ A^*P + PA \leq 0,\ PB = 0$" is infeasible iff all uncontrollable modes of (A, B) correspond to eigenvalues with positive real part
- "$P \neq 0,\ A^*P + PA \leq 0,\ PB = 0$" is infeasible iff (A, B) is controllable

Other results

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- "$A^*P + PA \lneqq 0$" is infeasible iff A is similar to a purely imaginary diagonal matrix
- "$A^*P + PA \leq 0$, $P \gneqq 0$" is infeasible iff $\lambda_i(A) \geq 0$ for all i
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- "$P \gneqq 0$, $A^*P + PA \leq 0$, $PB = 0$" is infeasible iff all uncontrollable modes of (A, B) correspond to eigenvalues with positive real part
- "$P \neq 0$, $A^*P + PA \leq 0$, $PB = 0$" is infeasible iff (A, B) is controllable

Frequency-domain inequalities: The KYP Lemma

Inequalities of the form

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} > 0$$

are commonly encountered in systems and control:

- Linear system analysis and design
- Digital filter design
- Robust control analysis
- Examples of constraints: $|H(j\omega)| < 1$ (small gain), $\Re H(j\omega) > 0$ (passivity), $H(j\omega) + H(j\omega)^* + H(j\omega)^* H(j\omega) < 1$ (mixed constraints)

The Kalman-Yakubovich-Popov Lemma

FDI

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} > 0$$

holds for all ω iff LMI

$$\begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} - M < 0$$

is feasible

- Infinite-dimensional constraint reduced to finite-dimensional constraint
- No sampling in frequency required

Control-theoretic proof of KYP Lemma

Suppose LMI

$$\begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} - M < 0$$

is feasible

Then

$$0 < \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \left(M - \begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} \right) \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}$$

$$= \begin{bmatrix} B^*(-j\omega I - A^*)^{-1} & I \end{bmatrix} M \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}$$

Control-theoretic proof of KYP Lemma

Suppose LMI

$$\begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} - M < 0$$

is feasible

Then

$$\begin{aligned} 0 &< \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* \left(M - \begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} \right) \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \\ &= \begin{bmatrix} B^*(-j\omega I - A^*)^{-1} & I \end{bmatrix} M \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \end{aligned}$$

Converse much harder; based on optimal control theory

New KYP lemma proof

More general version of the KYP Lemma:

Suppose $M_{22} > 0$

$$\begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} - M < 0,$$

is feasible iff

$$(j\omega I - A)u = Bv, \;\; (u, v) \neq 0 \Longrightarrow \begin{bmatrix} u^* & v^* \end{bmatrix} M \begin{bmatrix} u \\ v \end{bmatrix} > 0$$

- A can have imaginary eigenvalues
- If A has no imaginary eigenvalues, recover classical version

Duality-based KYP Lemma proof

Infeasibility of

$$\begin{bmatrix} A^*P + PA & PB \\ B^*P & 0 \end{bmatrix} - M < 0$$

equivalent to existence of Z s.t.

$$Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{bmatrix} \gneqq 0, \quad Z_{11}A^* + AZ_{11} + Z_{12}B^* + BZ_{12}^* = 0, \quad \mathbf{Tr} ZM \leq 0$$

- Must have $Z_{11} \gneqq 0$. Hence, factor Z as

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{bmatrix} = \begin{bmatrix} U & 0 \\ V & \hat{V} \end{bmatrix} \begin{bmatrix} U^* & V^* \\ 0 & \hat{V}^* \end{bmatrix},$$

where U has full rank

- Can show

$$US - AU = BV, \quad \textbf{Tr}\left(\begin{bmatrix} U^* & V^* \end{bmatrix} M \begin{bmatrix} U \\ V \end{bmatrix} \right) \leq 0,$$

  with $S + S^* = 0$

- Take Schur decomposition of S: $S = \sum_{i=1}^{m} j\omega_i q_i q_i^*$, with $\sum_i q_i q_i^* = I$

- Then

$$q_k^* \begin{bmatrix} U^* & V^* \end{bmatrix} M \begin{bmatrix} U \\ V \end{bmatrix} q_k \leq 0$$

  for some k

- Define $u = Uq_k$, $v = Vq_k$. Then

$$\begin{bmatrix} u^* & v^* \end{bmatrix} M \begin{bmatrix} u \\ v \end{bmatrix} \leq 0 \quad \text{and} \quad (j\omega I - A)u = Bv$$

Outline

- Theorem of alternatives for LMIs, and their applications
- SDP duality, and its application

Primal and dual SDPs

Primal SDP:

$$\begin{array}{ll} \texttt{minimize:} & \langle c, x\rangle_{\mathcal{V}} \\ \texttt{subject to:} & \mathcal{F}(x) + F_0 \geq 0 \end{array}$$

Dual SDP

$$\begin{array}{ll} \text{maximize} & -\langle F_0, Z\rangle_{\mathcal{S}} \\ \text{subject to} & \mathcal{F}^{\text{adj}}(Z) = c, \quad Z \geq 0 \end{array}$$

- If Z is dual feasible, then $-\mathbf{Tr} F_0 Z \leq p^*$
- If x is primal feasible, then $c^T x \geq d^*$
- Under mild conditions, $p^* = d^*$
- At optimum, $(\mathcal{F}(x_{\text{opt}}) + F_0)\, Z_{\text{opt}} = 0$

Application of duality: Bounds on H_∞ norm

Stable LTI system

$$\dot{x} = Ax + Bu, \;\; x(0) = 0, \quad y = Cx$$

- Transfer function $H(s) = C(sI - A)^{-1}B$
- $\mathbf{H}_\infty$ norm of H defined as

$$\|H\|_\infty = \sup_{\Re s > 0} \sigma_{\max}(H(s))$$

- $\|H\|_\infty^2$ equals maximum energy gain

$$\|H\|_\infty^2 = \max_u \frac{\int y^T y}{\int u^T u}$$

$\|H\|_\infty$ computation as an SDP

```
minimize:      
subject to:    
```

$$\begin{array}{ll} \text{minimize:} & \beta \\ \text{subject to:} & \begin{bmatrix} A^*P + PA + C^*C & PB \\ B^*P & -\beta I \end{bmatrix} \leq 0 \end{array}$$

$(\|H\|_\infty^2 = \beta_{\mathrm{opt}})$

Dual problem

$$\begin{array}{ll} \text{maximize:} & \mathbf{Tr}\, C Z_{11} C^* \\ \text{subject to:} & Z_{11}A^* + AZ_{11} + Z_{12}B^* + BZ_{12}^* = 0 \\ & \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{bmatrix} \geq 0, \quad \mathbf{Tr}\, Z_{22} = 1 \end{array}$$

Control-theoretic interpretation of dual problem

- Suppose $u(t)$ any input that steers state from $x(T_1) = 0$ to $x(T_2) = 0$, for some T_1, T_2. Let $y(t)$ be the corresponding output

- Define

$$Z_{11} = \int_{T_1}^{T_2} x(t)x(t)^* \, dt, \quad Z_{12} = \int_{T_1}^{T_2} x(t)u(t)^* \, dt, \quad Z_{22} = \int_{T_1}^{T_2} u(t)u(t)^* \, dt$$

Can show Z_{11}, Z_{12} and Z_{22} are dual feasible

- **Tr**$Z_{22} = \int_{T_1}^{T_2} u(t)^* u(t) \, dt = 1$ normalizes input energy

- Dual objective is corresponding output energy, gives lower bound:

$$\mathbf{Tr}\, C Z_{11} C^* = \int_{T_1}^{T_2} y(t)^* y(t) \, dt$$

New upper bounds on $\|H\|_\infty$

Recall primal problem:

$$\texttt{minimize:} \quad \beta$$
$$\texttt{subject to:} \quad \begin{bmatrix} A^*P + PA + C^*C & PB \\ B^*P & -\beta I \end{bmatrix} \le 0$$

A primal feasible point is

$$P = 2W_\mathrm{o}, \quad \beta = 4\lambda_{\max}(W_\mathrm{o}BB^*W_\mathrm{o}, C^*C)$$

where W_o is observability Gramian, obtained by solving $W_\mathrm{o}A + A^*W_\mathrm{o} + C^*C = 0$

Thus, new upper bound on $\|H\|_\infty$ is given by

$$2\sqrt{\lambda_{\max}(W_\mathrm{o}BB^*W_\mathrm{o}, C^*C)}$$

New lower bounds on $\|H\|_\infty$

Recall dual problem

$$\begin{array}{ll} \texttt{maximize:} & \mathbf{Tr}\, C Z_{11} C^* \\ \texttt{subject to:} & Z_{11}A^* + AZ_{11} + Z_{12}B^* + BZ_{12}^* = 0 \\ & \left[\begin{array}{cc} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{array}\right] \geq 0, \quad \mathbf{Tr}\, Z_{22} = 1 \end{array}$$

A dual feasible point is

$$Z_{11} = W_{\rm c}/\alpha, \quad Z_{12} = B/(2\alpha), \quad Z_{22} = B^* W_{\rm c}^{-1} B/(4\alpha),$$

where $\alpha = \mathbf{Tr}(B^* W_{\rm c}^{-1} B/4)$

Thus new lower bound is

$$2\sqrt{\mathbf{Tr}\, C W_{\rm c} C^* / (\mathbf{Tr}\, B^* W_{\rm c}^{-1} B)}$$

Application of duality: LQR problem

Primal

$$
\begin{array}{ll}
\texttt{minimize:} & \textbf{Tr} QZ_{11} + \textbf{Tr} Z_{22} \\
\texttt{subject to:} & AZ_{11} + BZ_{12}^* + Z_{11}A^* + Z_{12}B^* + x_0x_0^* \leq 0, \\
& \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{bmatrix} \geq 0
\end{array}
$$

Dual

$$
\begin{array}{ll}
\texttt{maximize:} & x_0^* P x_0 \\
\texttt{subject to:} & \begin{bmatrix} A^*P + PA + Q & PB \\ B^*P & I \end{bmatrix} \geq 0, \; P \geq 0
\end{array}
$$

The Linear-Quadratic Regulator problem

$$\dot{x} = Ax + Bu, \quad x(0) = x_0,$$

$$\text{find } u \text{ that minimizes } J = \int_0^\infty \left(x(t)^* Q x(t) + u(t)^* u(t) \right) \, dt,$$

s.t. $\lim_{t \to \infty} x(t) = 0$

Well-known solution: Solve Riccati equation

$$A^T P + PA + Q - PBB^T P = 0$$

such that $P > 0$. Then,

$$u_{\text{opt}}(t) = -B^T P x(t)$$

(Proof using quadratic optimal control theory)

Duality-based proof: Basic ideas

- Primal problem gives upper bound on LQR objective
- Dual problem gives lower bound on LQR objective
- Optimality condition gives LQR Riccati equation

Primal problem interpretation

Assume $u = Kx$, s.t. $x(t) \to 0$ as $t \to \infty$

Then LQR objective reduces to

$$J_K = \int_0^\infty x(t)^* \left(Q + K^*K\right) x(t)\, dt$$

and is an upper bound on the optimum LQR objective

- Condition $x(t) \to 0$ as $t \to \infty$ equivalent to

$$(A + BK)\tilde{Z} + \tilde{Z}(A + BK)^* + x_0 x_0^* = 0, \quad \tilde{Z} \geq 0$$

- LQR objective is **Tr**$\tilde{Z}(Q + K^*K)$

Best upper bound using state-feedback:

minimize: $\mathbf{Tr}\tilde{Z}(Q + K^*K)$
subject to: $\tilde{Z} \geq 0$
$(A + BK)\tilde{Z} + \tilde{Z}(A + BK)^* + x_0 x_0^* = 0$

With $Z_{11} = \tilde{Z}$, $Z_{12} = \tilde{Z}K^*$, $Z_{22} = K\tilde{Z}K^*$:

minimize: $\mathbf{Tr} QZ_{11} + \mathbf{Tr} Z_{22}$
subject to: $AZ_{11} + BZ_{12}^* + Z_{11}A^* + Z_{12}B^* + x_0 x_0^* \leq 0,$

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{bmatrix} \geq 0$$

(Same as primal problem)

Dual problem interpretation

Suppose for $P \geq 0$, $\frac{d}{dt}x(t)^*Px(t) \geq -\left(x(t)^*Qx(t) + u(t)^*u(t)\right)$, for all $t \geq 0$, and for all x and u satisfying $\dot{x} = Ax + Bu$, $x(T) = 0$. Then,

$$x_0^*Px_0 \leq \int_0^T \left(x(t)^*Qx(t) + u(t)^*u(t)\right)\ dt,$$

So $J_{\text{opt}} \geq x_0^*Px_0$

Derivative condition equivalent to LMI

$$\begin{bmatrix} A^*P + PA + Q & PB \\ B^*P & I \end{bmatrix} \geq 0$$

So lower bound to LQR objective given by dual problem

Optimality conditions

- Stabilizability of (A, B) guarantees strict primal feasibility
- Detectability of (Q, A) guarantees strict dual feasibility
- Recall, at optimality $(\mathcal{F}(x_{\rm opt}) + F_0)\, Z_{\rm opt} = 0$. This becomes

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{12}^* & Z_{22} \end{bmatrix} \begin{bmatrix} A^*P + PA + Q & PB \\ B^*P & I \end{bmatrix} = 0$$

Reduces to

$$\begin{bmatrix} I & K^* \end{bmatrix} \begin{bmatrix} A^*P + PA + Q & PB \\ B^*P & I \end{bmatrix} = 0,$$

or $K = -B^*P$, with all the eigenvalues of $A + BK$ having negative real part, and

$$A^*P + PA + Q - PBB^*P = 0$$

(Classical LQR result, derived using duality)

Conclusions

- SDP duality theory has interesting implications systems and control
- Implications for numerical computation:
  - ⋆ Dual problems sometimes have fewer variables
  - ⋆ Most efficient algorithms solve primal and dual together; control-theoretic interpretation can help increase efficiency