Improved Stability Analysis and Gain-Scheduled Controller Synthesis for Parameter-Dependent Systems*

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Abstract

We present new algorithms for robust stability analysis and gain-scheduled controller synthesis for linear systems affected by time-varying parametric uncertainties. These new techniques can also be applied to parameter-dependent nonlinear systems with real-rational nonlinearities. Sufficient conditions for robust stability as well as conditions for the existence of a robustly stabilizing gain-scheduled controller are given in terms of a finite number of Linear Matrix Inequalities (LMIs); explicit formulae for constructing robustly stabilizing gain-scheduled controllers are given in terms of the feasible set of these LMIs. The improvement offered by our approach over existing methods for stability analysis and gain-scheduled controller synthesis for parameter-dependent linear systems are analyzed in theory. Numerical examples demonstrate that our approach can offer significant improvement in practice.

1 Introduction

Our notation is standard. $\mathbf{R}^{m\times n}$ is the set of real $m\times n$ matrices. $\mathbf{C}^{m\times n}$ is the set of complex $m\times n$ matrices. $\mathbf{Co}\{\cdot\}$ denotes convex hull, that is,

$$\mathbf{Co}\{v_1, v_2, \ldots, v_n\} = \left\{ v \;\middle|\; v = \sum_{i=1}^{n} \lambda_i v_i,\ \sum_{i=1}^{n} \lambda_i \le 1, \lambda_i \ge 0 \right\}.$$

I_m is the $m\times m$ identity matrix; if its size can be determined from context, we will omit the subscript and simply denote it I. $P > 0$ means that P is a real, symmetric, positive-definite matrix. For a given set S and a positive number $\sigma > 0$, $\sigma S = \{\sigma s | s \in S\}$.

Consider the parameter-dependent system

$$\dot{x} = \mathbf{A}(\theta(t))x + \mathbf{B}(\theta(t))u, \qquad y = \mathbf{C}(\theta(t))x + \mathbf{D}(\theta(t))u, \tag{1}$$

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^{n_u}$ and $y(t) \in \mathbf{R}^{n_y}$, and $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are real-valued rational functions of the time-varying parameter vector $\theta(t) = [\theta_1(t) \cdots \theta_m(t)]^T \in \mathbf{R}^m$, which for all $t > 0$ is restricted to lie in a polytope $\mathbf{\Theta} \subset \mathbf{R}^m$ containing the origin. (When $\mathbf{\Theta}$ is not a polytope, the results developed herein can still be applied by replacing $\mathbf{\Theta}$ with some polytope $\mathbf{\Theta}_{\mathrm{poly}} \supset \mathbf{\Theta}$.) We assume that the function $\theta(\cdot)$ is such that the differential equation (1) has a solution. This technical admissibility

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condition will be implicit throughout the paper. For convenience, we will henceforth often drop the dependence of θ on t. The signals u and y have the interpretation of the control input and the measured output respectively. We assume that the parameters θ_i are unknown a priori, but can be measured in real-time, so that they can be incorporated, if possible, in a gain-scheduled control strategy. System (1) models a wide variety of commonly-encountered parameter-dependent systems; see, *e.g.*, [1, 2, 3].

We consider questions of stability analysis and stabilizing controller synthesis for system (1):

(P1) With u identically zero, does the state x of system (1) satisfy $\lim_{t\to\infty} x(t) = 0$ for every initial condition $x(0)$? If so, we say that system is robustly stable over $\mathbf{\Theta}$.

(P2) Does there exist a control law $u = K(y, \theta, t)$, such that the state x of system (1) satisfies $\lim_{t\to\infty} x(t) = 0$ for every initial condition $x(0)$? If so, we say that system is robustly stabilizable over $\mathbf{\Theta}$.

In addition, given an uncertainty set $\mathbf{\Theta}$, we define the *robust stability margin* of system (1) as

$$\sigma_{\mathrm{m}} = \sup\left\{\sigma \mid \text{for every } \gamma \in [0, \sigma], \text{ system (1) is robustly stable over } \gamma\mathbf{\Theta}\right\}$$

and the *robust stabilizability margin* as

$$\rho_{\mathrm{m}} = \sup\left\{\rho \mid \text{for every } \gamma \in [0, \rho], \text{ system (1) is robustly stabilizable over } \gamma\mathbf{\Theta}\right\}.$$

Let us first consider question (P1), that of robust stability over $\mathbf{\Theta}$. A number of numerically tractable sufficient conditions are available for robust stability, depending on the assumptions on the function $\mathbf{A}(\theta)$. One class of sufficient conditions is based on the notion of quadratic stability: The system $\dot{x} = \mathbf{A}(\theta)x$ is said to be quadratically stable if there exists a quadratic Lyapunov function $V(x) = x^T P x$ whose derivative is negative along every trajectory of the system, or equivalently, there exists $P = P^T$ such that

$$P > 0, \qquad P\mathbf{A}(\theta) + \mathbf{A}(\theta)^T P < 0 \text{ for all } \theta \in \mathbf{\Theta}. \tag{2}$$

For the simplest case when $\mathbf{A}(\theta)$ is an affine function of θ (this is the so-called *polytopic system* [4]), a necessary and sufficient condition for quadratic stability can be given in terms of a finite number of LMIs, one for each vertex of the polytope $\mathbf{\Theta}$ [5, 6]. For systems with more general uncertainties than polytopic ones, it is typical to consider a *linear fractional representation* (LFR) [7]. Here, the parameter-dependent system is represented as an LTI system, with the uncertain parameters appearing in a feedback loop as a diagonal uncertainty Δ; see Fig. 1. (We will henceforth refer to such systems as LFR systems.) Then, scaling matrices can be used in conjunction with the small-gain theorem to yield sufficient conditions for robust stability of system (1); see for example, [8, 9]. These scaling techniques can be reinterpreted in the more general framework of integral quadratic constraints (IQCs) [10, 11, 12, 13], and correspond to searching for more sophisticated Lyapunov functions [14].

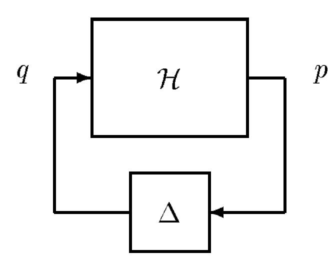


Figure 1: LFR of parameter-dependent uncertain systems.

The problem of controller synthesis (P2), has turned out to be considerably harder. While the problem of robustly stabilizing constant state-feedback synthesis for polytopic systems as well as LFR systems with real constant scalings has turned out to be convex, no convex reformulations are known for the simple problem of constant output feedback synthesis for even linear time-invariant systems. This makes the general output feedback controller synthesis problem very hard to solve. In many applications where the uncertain parameters θ can be measured in real-time [3, 15], the approach that holds the most promise for output feedback synthesis appears to be that of *gain-scheduled* controller synthesis: design feedback schemes that are themselves parameter-dependent on θ_i.

We note that our use of the term "gain-scheduled" refers to the linear parameter-varying and linear fractional transformation (LPV/LFT) approach for gain-scheduled control [16, 17, 18, 19, 20]. This is different from classical gain-scheduling controller synthesis techniques, where several controllers are designed under different operating conditions, with the actual control law switching between the locally designed controllers under some scheduling scheme [15]. A tutorial on gain-scheduled control techniques can be found in [19, 20].

Designing gain-scheduled output feedback controllers for polytopic systems or LFR systems using a single quadratic Lyapunov function can be reduced to finding the feasible set of a finite number of LMIs [17, 21, 22, 23]. Gain-scheduled controller synthesis for polytopic systems using parameter-dependent Lyapunov functions has also been studied in the literature; see for example [24, 18]. The parameter-dependent Lyapunov function is usually chosen to be of a special form, for example an affine [25] or a piecewise affine [26] function of the uncertainty. Although more general Lyapunov functions may yield less conservative analysis results, constructing such Lyapunov functions is much more difficult. Even for the special case when the Lyapunov function is an affince function of the uncertainty, one usually needs to grid the parameter space made up of the uncertainties and their first derivatives [24, 18, 27]; thus, in a sense, the result using finite gridding points (or finite number of LMIs) is unreliable. Although parameter gridding can be avoided in some cases, it requires either more restrictive assumptions on the system matrices [24], or the use of a more conservative cover for the set of uncertainties [28].

The approach that we employ in this paper begins with the following idea: For LFR systems, conventional robust stability analysis techniques require the search for structured scaling matrices that commute with the uncertainties; however, less conservative stability analysis results can be achieved by employing *unstructured* scaling matrices at different *vertices* of the parameter region. This idea can be regarded as an extension of the vertex-type stability analysis techniques in [5, 6], and has been proposed by Fu and Barabonov [29], and Iwasaki and Hara [30]. Our contributions, in this paper, are to extend this technique to the use of parameter-dependent Lyapunov functions, as well to gain-scheduled control synthesis. We will also show that the conventional constant scaling method can be viewed as a special case of our approach. Thus, our approach can effectively take advantage of the knowledge of polytopic covers that describe the uncertainties more accurately than conventional norm bounds, for both robustness analysis and gain-scheduled controller synthesis. In addition, the controller designed in our approach can be easily implemented in real-time.

The organization of the paper is as follows. In Section 2.1, we propose a vertex-type stability analysis technique using a quadratic Lyapunov function. We also discuss the relation between our vertex-type stability analysis results and the stability analysis techniques of Fu, Barabanov, Iwasaki and Hara [29, 30] as well as other quadratic stability conditions. In Section 2.2, we extend this approach to the search for a parameter-dependent Lyapunov function. In Section 3, we consider the gain-scheduled output feedback controller synthesis problem. In Section 4, we demonstrate through numerical examples that the robust stability analysis and gain-scheduled controller synthesis methods proposed herein offer significant improvement over existing constant scaling techniques. Proofs and technical details are in Appendices.

2 Robustness analysis using Lyapunov functions

2.1 Quadratic stability analysis

Consider the state-trajectories of system (1) with u identically zero:

$$\dot{x} = \mathbf{A}(\theta)x. \tag{3}$$

Since $\mathbf{A}(\theta)$ is a real-valued rational function of θ, we have $\mathbf{A}(\theta) = A + B_q\Delta(\theta)(I - D_{pq}\Delta(\theta))^{-1}C_p$ for some appropriate matrices[1] A, B_q, C_p and D_{pq}. Then, an equivalent linear-fractional representation [7, 16] of the autonomous system (3) is given by

$$\dot{x} = Ax + B_q q, \quad p = C_p x + D_{pq} q, \quad q = \Delta(\theta)p, \quad \Delta(\theta) = \mathbf{diag}(\theta_1 I_{s_1}, \ldots, \theta_m I_{s_m}), \tag{4}$$

where $x \in \mathbf{R}^n$, $q \in \mathbf{R}^d$, $p \in \mathbf{R}^d$ and A, B_q, C_p, D_{pq} are real matrices of appropriate sizes, with A being Hurwitz, i.e., with all its eigenvalues having negative real parts. The quantity $\max(s_1, \cdots, s_m)$ will be termed the *LFR degree* of system (4).

Define

$$\mathbf{\Delta} = \{\Delta(\theta) \mid \theta \in \mathbf{\Theta}\}. \tag{5}$$

Obviously $\mathbf{\Delta}$ is a polytope as well. Let Δ_i, $i = 1, \ldots, r$ be the vertices of $\mathbf{\Delta}$. The following theorem gives a sufficient condition for quadratic stability of system (4).

Theorem 2.1 *System (4) is quadratically stable if there exists $P \in \mathbf{R}^{n\times n}$ with $P = P^T > 0$ such that for every $\Delta \in \mathbf{\Delta}$, there exist $G_\Delta \in \mathbf{C}^{n\times d}$ and $H_\Delta \in \mathbf{C}^{d\times d}$ satisfying*

$$\begin{bmatrix} PA + A^TP + G_\Delta C_p + C_p^T G_\Delta{}^* & P(B_q\Delta) + G_\Delta(D_{pq}\Delta) - G_\Delta + C_p^T H_\Delta{}^* \\ (B_q\Delta)^T P + (D_{pq}\Delta)^T G_\Delta{}^* - G_\Delta{}^* + H_\Delta C_p & H_\Delta(D_{pq}\Delta) + (D_{pq}\Delta)^T H_\Delta{}^* - H_\Delta - H_\Delta{}^* \end{bmatrix} < 0. \tag{6}$$

Moreover, $V(\psi) = \psi^T P \psi$ is a Lyapunov function that proves quadratic stability of system (4).

Theorem 2.1 implies that the quadratic stability of system (4) can be established by checking condition (6) for all Δ in the polytope (5).

[1] Here we assume the LFR is well-posed, i.e., $\det(I - D_{pq}\Delta(\theta)) \neq 0$, $\forall \theta \in \mathbf{\Theta}$.

Remark 1 While a formal proof of Theorem 2.1 can be found in Appendix .1, we give an intuitive explanation behind it. Consider the block-diagram in Fig. 1. Conventional robustness analysis techniques take advantage of the structure and nature of Δ, for example by introducing scaling variables $\mathcal{D}$ and $\mathcal{D}^{-1}$ to the left and right of Δ. $\mathcal{D}$ is required to commute with Δ so that the solution to the closed-loop system equations remains unchanged. Thus, rather than $\mathcal{H}$, one now has available $\mathcal{D}^{-1} \circ \mathcal{H} \circ \mathcal{D}$ in the forward loop; the scaling variable $\mathcal{D}$ presents a new degree of freedom for improved robustness analysis.

Instead, consider Fig. 1 redrawn as in Fig. 2: The uncertainty Δ has been absorbed in the forward loop, making the identity operator I the "uncertainty". The simplicity of the feedback loop offers a much richer set of scaling variables $\mathcal{D}$; however, in the forward loop, we have $\mathcal{D}^{-1} \circ \mathcal{H} \circ \Delta \circ \mathcal{D}$; thus robust stability conditions are stated no longer in terms of $\mathcal{H}$ and $\mathcal{D}$, but in terms of Δ as well. The proof in Appendix .1, formalizes this discussion; in particular, see (39), which describes how the variables G_Δ and H_Δ arise. $\diamond$

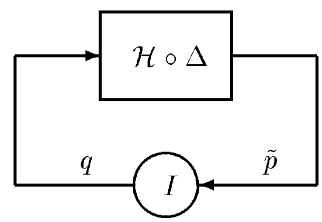


Figure 2: A loop transformation of the system in Fig. 1.

As Theorem 2.1 places no restrictions on G_Δ and H_Δ—these matrices may depend on Δ—it entails verifying an infinite number of matrix inequalities. This issue can be addressed by simply restricting G_Δ and H_Δ to be of special forms such that the left hand side of inequalities (6) is convex in Δ. It is then sufficient to check that inequality (6) holds for Δ_i, $i = 1, \ldots, r$, the vertices of the polytope (5). One such choice for G_Δ and H_Δ is described in the following Corollary; this choice is interesting in that it will prove useful in designing gain-scheduled output feedback controllers.

Corollary 2.2 *System (4) is quadratically stable if there exist $P = P^T > 0$ and $M = M^T > 0$ such that*

$$\begin{bmatrix} A^T P + PA + C_p^T M C_p & PB_{q,i} + C_p^T M D_{pq,i} \\ B_{q,i}^T P + D_{pq,i}^T M C_p & -M + D_{pq,i}^T M D_{pq,i} \end{bmatrix} < 0, \qquad i = 1 \ldots r, \tag{7}$$

where $B_{q,i} = B_q \Delta_i$ and $D_{pq,i} = D_{pq} \Delta_i$.

Remark 2 Corollary 2.2 can be viewed as an extension of the well-known structured scaling methods for robustness analysis. Consider the special case when $\Theta = [-\gamma, \gamma]^m$ with $\gamma > 0$, i.e., when it is a hypercube. In this case, structured scaling techniques [31, 32] can be shown to be equivalent to condition (7), with the additional restriction that $M = M^T > 0$ has such a structure that it commutes with Δ. (In our condition (7), M has no other constraints other than it being positive definite.) Also, our approach can effectively take advantage of the knowledge of polytopic covers that describe the uncertainties more accurately than conventional norm bounds.

While the quadratic stability condition (7) can be less conservative than the conventional structured scaling methods, we now have an increased number of optimization variables, owing to M being unstructured. In addition, the number of LMIs in our condition (7) equals r, the number of vertices of $\boldsymbol{\Delta}$. This reflects the added price to pay for the improvement in the robustness analysis [4]. $\diamond$

Other special cases for G_Δ and H_Δ are listed below:

1. Let $G_\Delta = G$ and $H_\Delta = H$ be any unstructured real constant matrices. Then (6) is feasible if

$$\begin{bmatrix} PA + A^T P + GC_p + C_p^T G^T & PB_{q,i} + GD_{pq,i} - G + C_p^T H^T \\ B_{q,i}^T P + D_{pq,i}^T G^T - G^T + HC_p & HD_{pq,i} + D_{pq,i}^T H^T - H - H^T \end{bmatrix} < 0, \quad i = 1, \dots, r. \qquad (8)$$

2. Let $G_\Delta = C_p^T M/2$ and $H_\Delta = ((D_{pq}\Delta)^T + I)M/2 + \Delta^T S^*$, with $\Delta = \Delta^T$, $M = M^T > 0$ being a real constant matrix and $S = -S^*$, and with both M and S commuting with Δ. Then (6) is feasible if

$$\begin{bmatrix} A^T P + PA + C_p^T M C_p & PB_q\Delta + C_p^T M D_{pq}\Delta + C_p^T S\Delta \\ \Delta^T B_q^T P + \Delta^T D_{pq}^T M C_p - \Delta^T S C_p & \begin{matrix} -M + \Delta^T D_{pq}^T M D_{pq}\Delta \\ +\Delta^T D_{pq}^T S\Delta - \Delta^T S D_{pq}\Delta \end{matrix} \end{bmatrix} < 0. \qquad (9)$$

If $S = -S^T$ is real and skew-symmetric, (9) yields the stability criterion derived in [16], which is also the stability criterion for systems affected by time-varying real uncertainties, obtained in the IQC framework [10]. If $S = jN$ with $N = N^T$ being real and symmetric, (9) gives the constant-scaling version of the the stability criterion from real-μ analysis [9].

3. Let $G_\Delta = (QC_p)^T/2$ and $H_\Delta = (2S\Delta + QD_{pq}\Delta + Q)^T/2$, where $Q = Q^T$ and S are real matrices. Then (6) is feasible if there exists $M = M^T$ such that

$$\begin{gathered} \begin{bmatrix} A^T P + PA & PB_q \\ B_q^T P & 0 \end{bmatrix} + \begin{bmatrix} 0 & C_p^T \\ I & D_{pq}^T \end{bmatrix} \begin{bmatrix} M & S^T \\ S & Q \end{bmatrix} \begin{bmatrix} 0 & I \\ C_p & D_{pq} \end{bmatrix} < 0, \\ \begin{bmatrix} \Delta \\ I \end{bmatrix}^T \begin{bmatrix} M & S^T \\ S & Q \end{bmatrix} \begin{bmatrix} \Delta \\ I \end{bmatrix} \geq 0. \end{gathered} \qquad (10)$$

Condition (10) is the same as the stability criterion derived in [29, 30], or by using IQC approach with time-invariant multipliers [10], or by using full block scalings[2] [33]. We note that condition (10) is in general less conservative than condition (7), in which $S = 0$ and $M = -Q$.

4. Consider the special case when $D_{pq} = 0$ in system (4). Then the robust stability condition (10) is equivalent to

$$\begin{bmatrix} A^T P + PA + C_p^T Q C_p & PB_{q,i} + C_p^T S_i \\ B_{q,i}^T P + S_i^T C_p & -(Q + S_i + S_i^T) \end{bmatrix} < 0, \quad i = 1 \dots r, \qquad (11)$$

where $S_i = S\Delta_i$. Condition (11) is necessary and sufficient for quadratic stability of the system $\dot{x} = (A + B_q\Delta(\theta)C_p)x$ over $\boldsymbol{\Theta}$, see [10]. Indeed in this case, we can always define $S_i = -Q = I$ in (11) without loss of generality. (See Appendix B.)

[2]The stability condition (10) based on full block scalings is applied in [33, 34] to solve gain-scheduled controller synthesis problems. However as pointed out in [34], the procedure to construct the resulting controllers is numerically delicate.

Remark 3 Further improvement of condition (7) is possible. Let $G_\Delta = C_p^T M_\Delta/2$ and $H_\Delta = ((D_{pq}\Delta)^T + I)M_\Delta/2$, where M_Δ depends on Δ. Then, (6) is equivalent to

$$\begin{bmatrix} A^TP + PA + C_p^T M_\Delta C_p & P(B_q\Delta) + C_p^T M_\Delta (D_{pq}\Delta) \\ (B_q\Delta)^T P + (D_{pq}\Delta)^T M_\Delta C_p & -M_\Delta + (D_{pq}\Delta)^T M_\Delta (D_{pq}\Delta) \end{bmatrix} < 0, \quad \Delta \in \mathbf{Co}\{\Delta_1, \ldots, \Delta_r\}. \quad (12)$$

Since M_Δ is a variable that depends on Δ, condition (12) is less conservative than condition (7). While condition (12) is not tractable in general as it includes infinitely many inequalities, it may be applied in robust stability analysis as well as in gain-scheduled controller synthesis by dividing the uncertain parameter set $\mathbf{\Theta}$ into several smaller sets and employing a different M_Δ for each smaller set. However, the improvement of the analysis and synthesis results comes at the added expense of more LMIs. ◇

For a special case when the LFR degree of system (4) is one, condition (12) is equivalent to checking a finite number of LMIs, as the following shows.

Corollary 2.3 *Suppose the LFR degree of system (4) is one. If there exist* $P = P^T > 0$ *and* $M_i = M_i^T > 0$ *such that*

$$\begin{bmatrix} A^TP + PA + C_p^T M_i C_p & PB_{q,i} + C_p^T M_i D_{pq,i} \\ B_{q,i}^T P + D_{pq,i}^T M_i C_p & -M_i + D_{pq,i}^T M_i D_{pq,i} \end{bmatrix} < 0, \quad i = 1, \ldots, r, \quad (13)$$

where $B_{q,i} = B_q\Delta_i$ *and* $D_{pq,i} = D_{pq}\Delta_i$, Δ_i *is a vertex of the polytope (5), then system (4) is quadratically stable.*

Condition (13) is clearly less stringent than condition (7), as it allows for different scaling matrices M_i for different vertices. We will therefore refer to the application of Corollary 2.3 as a vertex-dependent scaling method or simply "vertex scaling".

To conclude this section, we note that while all the stability conditions in this section are based on the parametrization of the quadratic Lyapunov function as $V(\psi) = \psi^T P \psi$, it is straightforward to re-derive the results with the Lyapunov function parametrized as $V(\psi) = \psi^T Q^{-1} \psi$.

2.2 Robustness analysis using parameter-dependent Lyapunov functions

In Section 2.1, the robustness analysis is based on a single quadratic Lyapunov function. If system (3) is affinely parameter-dependent or polytopic, parameter-dependent Lyapunov functions can be employed, yielding improved robust stability analysis results [25]. Specifically, if the rate of time-variation of the uncertainty θ is bounded and lies in the set $\mathbf{\Phi}$ ($\mathbf{\Phi} = \{0\}$ corresponds to the case of a real constant uncertainty), it has been shown in [25] that the system $\dot{x} = \mathbf{A}(\theta)x$, where $\mathbf{A}(\theta)$ is an affine function of θ, is stable if there exists a parameter-dependent Lyapunov function $V(x) = x^T P(\theta) x$ such that for all $\theta \in \mathbf{\Theta}$,

$$P(\theta) > 0, \quad \dot{P}(\theta) + P(\theta)\mathbf{A}(\theta) + \mathbf{A}(\theta)^T P(\theta) < 0. \quad (14)$$

Since the knowledge of the time varying rate of the uncertainty is incorporated in the analysis, condition (14) is evidently less conservative than condition (2). However, condition (14) consists of an infinite number of inequalities, even in the simple case when $\mathbf{A}(\theta)$ and $P(\theta)$ are restricted to be affine functions of θ. Multi-convexity techniques are proposed in [25, 35, 36] in order to derive a sufficient condition consisting of a finite number of LMIs for (14) to hold.

The robust stability condition (14) has been applied to design gain-scheduled controllers for affine or piecewise affine parametric uncertain systems using multi-convexity techniques [24, 36, 37], or using a more conservative cover of the set of uncertainties [28].

In this paper, we consider the robust stability analysis and gain-scheduled controller design problems for LFR systems. For LFR systems with time-invariant parametric uncertainties, Fu and Dasgupta [38] have proposed to solve the stability analysis problems using a parameter-dependent Lyapunov function with parameter-dependent multipliers. In this paper, we consider LFR systems with time-varying parametric uncertainties. Our approach is to combine polytopic system analysis methods with conventional constant scaling techniques to solve the robustness analysis problem in LFR uncertain systems. Perhaps more important, we show that this analysis approach can be extended to design gain scheduled controllers using a parameter-dependent Lyapunov function.

Following the same line as Theorem 2.1, in this section we derive a sufficient condition for robust stability using parameter-dependent Lyapunov functions, summarized in the following corollary.

Corollary 2.4 *Let* $\Theta_i = [\Theta_{i1} \cdots \Theta_{im}]^T$, $i = 1, \ldots, r$ *be the vertices of* $\mathbf{\Theta}$ *and* $\Delta_i = \Delta(\Theta_i)$. *Let* $\Phi_k = [\Phi_{k1} \cdots \Phi_{km}]^T$, $k = 1, \ldots, v$ *be the vertices of* $\mathbf{\Phi}$. *System (4) is stable for all* θ *satisfying* $(\theta, \dot{\theta}) \in \mathbf{\Theta} \times \mathbf{\Phi}$ *if there exist* $Q_j = Q_j^T \in \mathbf{R}^{n \times n}$ *and* $M = M^T > 0$ *such that*

$$\begin{bmatrix} E_{11,ik} & E_{12,i} \\ E_{12,i}^T & E_{22,i} \end{bmatrix} < 0 \quad \textit{and} \quad Q_0 + \sum_{j=1}^{m} \Theta_{ij} Q_j > 0, \quad i = 1, \ldots, r, \quad k = 1, \ldots, v, \tag{15}$$

where

$$E_{11,ik} = A \left(Q_0 + \sum_{j=1}^{m} \Theta_{ij} Q_j \right) + \left(Q_0 + \sum_{j=1}^{m} \Theta_{ij} Q_j \right) A^T + B_{q,i} M B_{q,i}^T - \sum_{j=1}^{m} \Phi_{kj} Q_j,$$

$$E_{12,i} = \left(Q_0 + \sum_{j=1}^{m} \Theta_{ij} Q_j \right) C_p^T + B_{q,i} M D_{pq,i}^T, \quad E_{22,i} = -M + D_{pq,i} M D_{pq,i}^T,$$

and $B_{q,i} = B_q \Delta_i$, $D_{pq,i} = D_{pq} \Delta_i$, $i = 1 \ldots r$, $k = 1 \ldots v$.

Remark 4 When $Q_j = 0$, $j = 1, \ldots, m$, condition (15) reduces to the robust stability conditions based on a single quadratic Lyapunov function, given in Corollary 2.2, where the information on the rate of variation of the uncertainty is not incorporated into the stability analysis. (The variables Q and M in Corollary 2.4 correspond to the variables P^{-1} and M^{-1} respectively in Corollary 2.2.) Thus Corollary 2.4 offers the potential for improved stability analysis when information on the rate of variation of the uncertainty is available. However, this comes with the added price that an LMI needs to be checked at every vertex of $\mathbf{\Theta} \times \mathbf{\Phi}$. $\diamond$

Remark 5 While Corollary 2.4 applies to the more general LFR systems, it can also be used for robustness analysis of polytopic systems. The resulting robust stability condition consists of a finite number of LMIs, as compared to an infinite number of LMIs that result from the approach in [25]. However, we would expect the latter condition to be less conservative. ◇

If the set $\mathbf{\Theta}\times\mathbf{\Phi}$ is symmetric around the origin, i.e., $(\theta,\dot{\theta})\in\mathbf{\Theta}\times\mathbf{\Phi}$ implies $(-\theta,-\dot{\theta})\in\mathbf{\Theta}\times\mathbf{\Phi}$, it can be verified that if condition (15) is feasible for some Q_i, then the system must be quadratically stable, i.e., $Q_i = 0, i = 1,2,\ldots,m$. Therefore in this case Corollary 2.4 offers no improvement over quadratic stability. However, for cases when set of uncertainties is not symmetric around the origin, Corollary 2.4 can be strictly less conservative than the quadratic stability condition (see the numerical example in Section 4.3).

Note that Corollary 2.4 offers an approach for the direct use of parameter-dependent Lyapunov functions in the analysis of systems that exhibit a rational dependence on the parameters. This is in contrast to indirect techniques such as using a (conservative) polytopic cover for

$$\left\{(\theta,\dot{\theta},\theta_1\theta_1,\ldots\theta_i\theta_j,\ldots,\theta_m\theta_m)\,\middle|\,\theta\in\mathbf{\Theta},\dot{\theta}\in\mathbf{\Phi},i\le j\right\}$$

as in [35, 28], or using a multi-convexity method [24, 25, 36].

3 Gain-scheduled output feedback synthesis

We next consider the problem of designing a gain-scheduled output feedback control strategy $u = K(y,\theta)$ such that system (1) is robustly stable. In particular, we show that the sufficient condition for robust stability that we stated in Corollary 2.2 and Corollary 2.4 for system (3) can be directly extended to designing a gain-scheduled controller $K(y,\theta)$ that is guaranteed to stabilize system (1). As noted in Section 2, our analysis technique guarantees a larger stability margin over conventional constant structured scaling methods; therefore, the corresponding gain-scheduled controller will come with a larger guaranteed closed-loop stability margin as well.

Consider system (1). Since $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ are real-valued rational functions of θ, we have

$$\begin{bmatrix}\mathbf{A}(\theta) & \mathbf{B}(\theta)\\ \mathbf{C}(\theta) & \mathbf{D}(\theta)\end{bmatrix} = \begin{bmatrix}A & B_u\\ C_y & D_{yu}\end{bmatrix} + \begin{bmatrix}B_q\\ D_{yq}\end{bmatrix}\Delta(\theta)(I-D_{pq}\Delta(\theta))^{-1}\begin{bmatrix}C_p & D_{pu}\end{bmatrix} \tag{16}$$

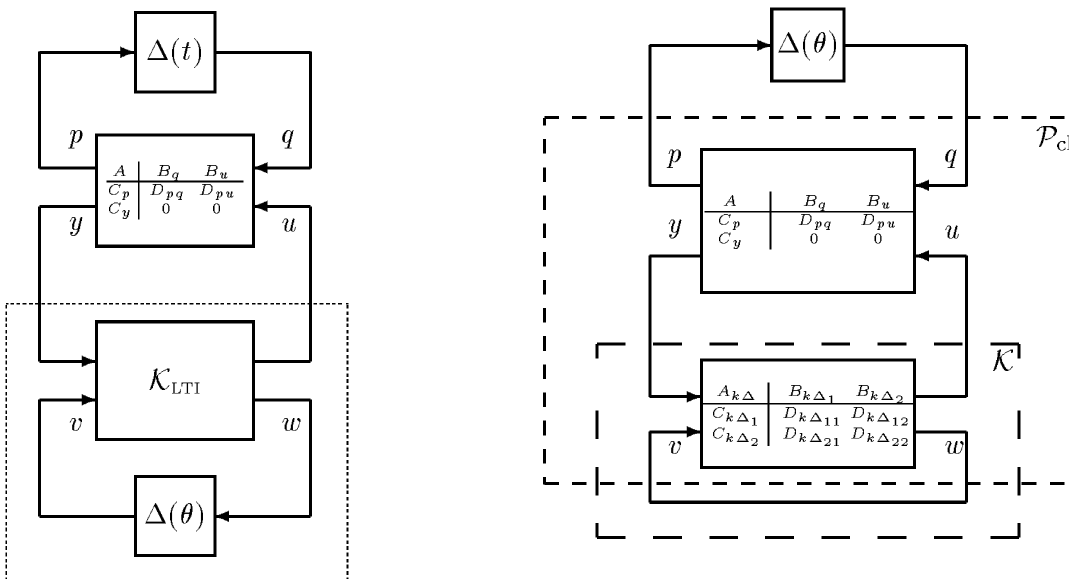
for some appropriate matrices A, B_q, B_u, C_p, C_y, D_{yq}, D_{yu}, D_{pq} and D_{pu}. Then, an equivalent linear-fractional representation of system (1) is described by

$$\begin{gathered}\dot{x} = Ax + B_q q + B_u u,\quad p = C_p x + D_{pq} q + D_{pu} u,\quad y = C_y x + D_{yq} q + D_{yu} u.\\ q = \Delta(\theta)p,\quad \Delta(\theta) = \mathbf{diag}(\theta_1 I_{s_1},\ldots,\theta_m I_{s_m}),\end{gathered} \tag{17}$$

where $x\in\mathbf{R}^n$, $q\in\mathbf{R}^d$, $p\in\mathbf{R}^d$, $u\in\mathbf{R}^{n_u}$ and $y\in\mathbf{R}^{n_y}$. We will henceforth assume that $D_{yu}=0$ and $D_{yq}=0$. The former is a standard assumption and can always be satisfied via loop transformations, while the latter is a technical assumption that implies that there is no uncertainty in the measured output.

The new gain-scheduled control scheme is shown in Fig. 3(b). Comparing it to the conventional gain-scheduled control scheme shown in Fig. 3(a) (see [21]), two differences are apparent. The first is that with our scheme, every system matrix is scheduled. The second difference is that our controller includes a unity feedback. This, while causing no loss of generality, will be important in establishing LMI conditions for the existence of a stabilizing gain-scheduled controller. Our gain-scheduled controller design procedure is similar to the one used in [21], with the important difference that the scalings that we employ are unstructured.

Remark 6 The intuitive reason behind the controller architecture in Fig. 3(b) is similar to the one outlined in Remark 1. Recall that with our analysis techniques, the plant has the uncertainty Δ absorbed in it, resulting in unity feedback. It turns out in order to carry through the gain-scheduled feedback synthesis procedure, the controller must have the same structure, hence a unity feedback architecture. $\diamond$



(a) Conventional gain-scheduled output feedback controller.

(b) New gain-scheduled output feedback controller.

Figure 3: Gain-scheduled control strategies.

The gain-scheduled controller $\mathcal{K}$ can be described by the state equations

$$\begin{array}{rclrcl} \dot{x}_k &=& A_{k\Delta}x_k + B_{k\Delta_1}y + B_{k\Delta_2}v, & u &=& C_{k\Delta_1}x_k + D_{k\Delta_{11}}y + D_{k\Delta_{12}}v, \\ w &=& C_{k\Delta_2}x_k + D_{k\Delta_{21}}y + D_{k\Delta_{22}}v, & v &=& w, \end{array} \tag{18}$$

where $x_k(t) \in \mathbf{R}^{n_k}$, $v \in \mathbf{R}^d$ and $w \in \mathbf{R}^d$.

All the state-space matrices in (18) are functions of the time-varying matrix $\Delta(\theta)$; hence their subscript Δ. (Their exact dependence on Δ will become clear later.) Then, with the notation

$$\Omega(\theta) = \begin{bmatrix} A_{k\Delta} & B_{k\Delta_1} & B_{k\Delta_2} \\ C_{k\Delta_1} & D_{k\Delta_{11}} & D_{k\Delta_{12}} \\ C_{k\Delta_2} & D_{k\Delta_{21}} & D_{k\Delta_{22}} \end{bmatrix} \tag{19}$$

the state space equations governing the closed-loop system $\mathcal{P}_{\rm cl}$ are

$$\begin{bmatrix} \dot{x} \\ \dot{x}_k \end{bmatrix} = A_{\rm cl}(\theta) \begin{bmatrix} x \\ x_k \end{bmatrix} + B_{\rm cl}(\theta) \begin{bmatrix} v \\ q \end{bmatrix}, \quad \begin{bmatrix} w \\ p \end{bmatrix} = C_{\rm cl}(\theta) \begin{bmatrix} x \\ x_k \end{bmatrix} + D_{\rm cl}(\theta) \begin{bmatrix} v \\ q \end{bmatrix},$$

$$\begin{bmatrix} v \\ q \end{bmatrix} = \begin{bmatrix} I & \\ & \Delta(\theta) \end{bmatrix} \begin{bmatrix} w \\ p \end{bmatrix},$$

where

$$\begin{aligned} A_{\rm cl}(\theta) &= \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} + \mathcal{B}\Omega(\theta)\mathcal{C}, \quad B_{\rm cl}(\theta) = \begin{bmatrix} 0 & B_q \\ 0 & 0 \end{bmatrix} + \mathcal{B}\Omega(\theta)\mathcal{D}_{yq}, \\ C_{\rm cl}(\theta) &= \begin{bmatrix} 0 & 0 \\ C_p & 0 \end{bmatrix} + \mathcal{D}_{pu}\Omega(\theta)\mathcal{C}, \quad D_{\rm cl}(\theta) = \begin{bmatrix} 0 & 0 \\ 0 & D_{pq} \end{bmatrix} + \mathcal{D}_{pu}\Omega(\theta)\mathcal{D}_{yq}, \end{aligned} \tag{20}$$

with

$$\mathcal{B} = \begin{bmatrix} 0 & B_u & 0 \\ I & 0 & 0 \end{bmatrix}, \quad \mathcal{D}_{pu} = \begin{bmatrix} 0 & 0 & I \\ 0 & D_{pu} & 0 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & I \\ C_y & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathcal{D}_{yq} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \end{bmatrix}. \tag{21}$$

3.1 Gain-scheduled controller design using quadratic Lyapunov functions

We first design quadratically stabilizing gain-scheduled output feedback controllers by applying Corollary 2.2. The following theorem provides a sufficient condition for the existence of a full order robustly stabilizing gain-scheduled controller.

Theorem 3.1 *Consider the closed-loop system in Fig. 3(b). Let $n_k = n$. Then, given $\gamma > 0$, there exist $P = P^T > 0$ and $M = M^T > 0$ such that for every $\theta \in \gamma\mathbf{\Theta}$, there exists $\Omega(\theta)$ satisfying*

$$\begin{bmatrix} A_{\rm cl}^T(\theta)P + PA_{\rm cl}(\theta) + C_{\rm cl}^T(\theta)MC_{\rm cl}(\theta) & PB_{\rm cl}(\theta) + C_{\rm cl}^T(\theta)MD_{\rm cl}(\theta) \\ B_{\rm cl}^T(\theta)P + D_{\rm cl}^T(\theta)MC_{\rm cl}(\theta) & -M + D_{\rm cl}^T(\theta)MD_{\rm cl}(\theta) \end{bmatrix} < 0, \tag{22}$$

if and only if there exist $R \in \mathbf{R}^{n\times n}$ and $S \in \mathbf{R}^{n\times n}$, $L \in \boldsymbol{R}^{d\times d}$ and $J \in \boldsymbol{R}^{d\times d}$ such that the following matrix inequalities hold:

$$\begin{aligned} &\begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} AR + RA^T & RC_p^T & \gamma B_{q,i}J \\ C_pR & -J & \gamma D_{pq,i}J \\ \hline \gamma JB_{q,i}^T & \gamma JD_{pq,i}^T & -J \end{array}\right] \begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix} < 0, \\ &\begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} A^TS + SA & \gamma SB_{q,i} & C_p^TL \\ \gamma B_{q,i}^TS & -L & \gamma D_{pq,i}^TL \\ \hline LC_p & \gamma LD_{pq,i} & -L \end{array}\right] \begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix} < 0, \end{aligned} \tag{23a}$$

$$\begin{bmatrix} S & I \\ I & R \end{bmatrix} > 0, \quad \begin{bmatrix} L & I \\ I & J \end{bmatrix} > 0, \quad i = 1, \ldots, r. \tag{23b}$$

where $B_{q,i} = B_q\Delta_i$, $D_{pq,i} = D_{pq}\Delta_i$, Δ_i, $i = 1, \ldots, r$ are the vertices of the polytope $\mathbf{\Delta}$, N_R and N_S are matrices whose columns comprise the bases of the null spaces of $[B_u^T \quad D_{pu}^T]$ and $[C_y \quad 0]$ respectively.

The main implication of Theorem 3.1 is that we now have a sufficient condition for the existence of a robustly stabilizing gain-scheduled controller for system (1). In contrast with the gain-scheduled controller designed in [21] and [16], there are no structure constraints on M in Theorem 3.1; consequently, even in the case when $\mathbf{\Theta}$ is a hypercube, our design is at most as conservative as the design using structured scalings. Of course, as with the stability analysis, our design can also directly take advantage of the knowledge of more accurate polytopic covers for the uncertainties.

Remark 7 Note that since the gain-scheduled controller depends on the uncertain parameters θ, the LFR degree of the closed loop system is always greater than one. Thus we may not use vertex scaling (Corollary 2.3) in designing gain-scheduled controllers. However for general output feedback synthesis problems (see [39]), we may still apply Corollary 2.3 to design output feedback controllers that guarantee larger closed-loop stability margins. $\diamond$

A direct consequence of Theorem 3.1 is that a lower bound of the robust stabilizability margin $\rho_{\rm m}$ can be computed by solving the following Generalized Eigenvalue Minimization Problem (GEVP).

`Minimize:` κ

`Subject to:`

$$\begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} AR+RA^T & RC_p^T & B_{q,i}J \\ C_pR & -X & D_{pq,i}J \\ \hline JB_{q,i}^T & JD_{pq,i}^T & -X \end{array}\right] \begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix} < 0,$$

$$\begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} A^TS+SA & SB_{q,i} & C_p^TL \\ B_{q,i}^TS & -Y & D_{pq,i}^TL \\ \hline LC_p & LD_{pq,i} & -Y \end{array}\right] \begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix} < 0, \qquad (24)$$

$$\begin{bmatrix} S & I \\ I & R \end{bmatrix} > 0, \quad \begin{bmatrix} L & I \\ I & J \end{bmatrix} > 0, \quad X < \kappa J, \quad Y < \kappa L \quad i = 1, \ldots, r.$$

where $B_{q,i} = B\Delta_i$, $D_{pq,i} = D_{pq}\Delta_i$, X, Y, R, S, L, and J are optimization variables, $\rho_{\rm m} = 1/\kappa$ is the robustly stabilizability margin.

For a given $\gamma > 0$, we have thus far only derived conditions for the existence of a quadratically stabilizing gain-scheduled output feedback controller over $\gamma\mathbf{\Theta}$. We now describe an algorithm for explicitly constructing a family of gain-scheduled output feedback controllers that are guaranteed to stabilize the system over $\gamma\mathbf{\Theta}$.

Step 1. Design controllers corresponding to each vertex of the polytope $\gamma\mathbf{\Delta}$ *.*

In the first step, we apply a similar technique as the gain-scheduled controller synthesis algorithm in [21], and we therefore refer to it for details.

Let $(R,\ S,\ L,\ J)$ be a feasible solution to (23). From the proof of Theorem 3.1, S and R are the upper left blocks on the diagonal of the matrices P and P^{-1} respectively; L and J are the lower right blocks on the diagonal of the scaling matrices M and M^{-1} respectively. Noting that $S > R^{-1}$, we define $P_{12} = (S - R^{-1})^{1/2}$ and $Q_{12} = -RP_{12}$, and

$$P = \begin{bmatrix} S & P_{12} \\ P_{12}^T & I \end{bmatrix}. \qquad \text{Then,} \quad P^{-1} = \begin{bmatrix} R & Q_{12} \\ Q_{12}^T & I - P_{12}^TQ_{12} \end{bmatrix}. \qquad (25)$$

Next, with $M_{12} = (L - J^{-1})^{1/2}$ and $N_{12} = -JM_{12}$, define

$$M = \begin{bmatrix} I & M_{12}^T \\ M_{12} & L \end{bmatrix}. \qquad \text{Then,} \quad M^{-1} = \begin{bmatrix} I - M_{12}^T N_{12} & N_{12}^T \\ N_{12} & J \end{bmatrix}.$$

With P, M and M^{-1} defined above, for each Δ_i, $i = 1, \ldots, r$, the LMI (42) must be feasible from the equivalence between (42) and (23). By solving the LMI feasibility problem (42) for $\Omega_{\Delta_i} = \Omega(\Delta_i(\theta))$, we get a controller Ω_{Δ_i} corresponding to the vertex Δ_i.

Step 2. Design the gain-scheduled controller.

For any $\Delta(\theta) \in \gamma\boldsymbol{\Delta}$, solve the set of linear equations $\Delta(\theta) = \sum_{i=1}^{r} \alpha_i(\theta)\gamma\Delta_i$ to get $\alpha_i(\theta)$. Define

$$\Omega(\theta) = \sum_{i=1}^{r} \alpha_i(\theta)\gamma\Omega_{\Delta_i}. \tag{26}$$

Then, (26) gives the state space matrices of a gain-scheduled controller that is guaranteed to quadratically stabilize the system.

Note that in controller (26), every system matrix of the controller in (19) is scheduled, that is, depends on θ (unlike with [21, 16] where only B_{k_2}, $D_{k_{12}}$ and $D_{k_{22}}$ are scheduled).

3.2 Gain-scheduled controller design using parameter-dependent Lyapunov functions

For affinely or piecewise affinely parameter-dependent systems, the gain-scheduled controller design based on a single quadratic Lyapunov function can be improved by using parameter-dependent Lyapunov functions [24, 37]. In this section, we extend the use of parameter-dependent Lyapunov functions for designing gain-scheduled controllers for more general parameter-dependent systems described by the LFR framework.

Theorem 3.2 *Consider the parameter-dependent system with a full order output feedback controller in Fig. 3(b), i.e., $n_k = n$. Then the parameter-dependent system is robustly stabilizable for all θ satisfying $(\theta, \dot{\theta}) \in \boldsymbol{\Theta} \times \boldsymbol{\Phi}$ if there exist $R_j \in \mathbf{R}^{n\times n}$, $S_j \in \mathbf{R}^{n\times n}$, $L \in \boldsymbol{R}^{d\times d}$, $J \in \boldsymbol{R}^{d\times d}$ and $X \in \boldsymbol{R}^{d\times d}$ such that the following matrix inequalities hold:*

$$\begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} E_{11} & E_{12} & E_{13} \\ E_{12}^T & -J & E_{23} \\ \hline E_{13}^T & E_{23}^T & -J \end{array}\right] \begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix} < 0, \tag{27a}$$

$$\begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} F_{11} & F_{12} & F_{13} \\ F_{12}^T & -X & F_{23} \\ \hline F_{13}^T & F_{23}^T & -L \end{array}\right] \begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix} < 0, \tag{27b}$$

$$\begin{bmatrix} S & I \\ I & R \end{bmatrix} > 0, \quad \begin{bmatrix} L & I \\ I & J \end{bmatrix} > 0, \quad \begin{bmatrix} L & \Delta(\theta)X \\ X\Delta(\theta) & X \end{bmatrix} > 0, \tag{27c}$$

where

$$R = R_0 + \sum_{j=1}^{m} \theta_j(t) R_j, \quad S = S_0 + \sum_{j=1}^{m} \theta_j(t) S_j,$$

$$E_{11} = AR + RA^T - \sum_{j=1}^{m} \dot{\theta}_j(t) R_j, \quad E_{12} = RC_p^T, \quad E_{13} = B_q(\theta)J, \quad E_{23} = D_{pq}(\theta)J,$$

$$F_{11} = A^T S + SA + \sum_{j=1}^{m} \dot{\theta}_j(t) S_j, \quad F_{12} = SB_q, \;\; F_{13} = C_p^T L, \;\; F_{23} = D_{pq}^T L.$$

$B_q(\theta) = B_q \Delta(\theta)$, $D_{pq}(\theta) = D_{pq}\Delta(\theta)$, and N_R and N_S are matrices whose columns comprise the bases of the null spaces of $[B_u^T \;\; D_{pu}^T]$ and $[C_y \;\; 0]$ respectively.

Note that conditions (27) are linear matrix inequalities. Therefore we only need to check the feasibility of these LMIs on the vertices Δ_i and Φ_k. Moreover, when $R_j = 0$ and $S_j = 0$, $j = 1, \ldots, m$, conditions (27) yield the quadratically stabilizing gain-scheduled controller synthesis condition (23); this corresponds to the situation when the rate of variation of the uncertainty cannot be measured in real time or is unbounded. Conditions (27) offer the potential for improved gain-scheduled controller design when information on the rate of variation of the uncertainty is available.

We now present an algorithm for explicitly constructing a family of gain-scheduled output feedback controllers that are guaranteed to stabilize the system. Suppose that the synthesis conditions (27) are feasible for some L, J, and R_j, S_j, $j = 1, \ldots, m$.

Step 1. With the measured $(\theta, \dot{\theta}) \in \mathbf{\Theta} \times \mathbf{\Phi}$, construct $P(\theta)$ and $P(\theta)^{-1}$.

Let

$$S = S_0 + \sum_{j=1}^{m} \theta_j(t) S_j, \quad R = R_0 + \sum_{j=1}^{m} \theta_j(t) R_j. \qquad \text{Then, } \dot{S} = \sum_{j=1}^{m} \dot{\theta}_j(t) S_j, \quad R = \sum_{j=1}^{m} \dot{\theta}_j(t) R_j.$$

Define

$$P(\theta) = \begin{bmatrix} S & -(S - R^{-1}) \\ -(S - R^{-1}) & S - R^{-1} \end{bmatrix}, \quad \text{and} \quad M = \begin{bmatrix} I & M_{12}^T \\ M_{12} & L \end{bmatrix} > 0, \tag{28}$$

where $M_{12} = (L - J^{-1})^{1/2}$. Thus

$$P(\theta)^{-1} = \begin{bmatrix} R & R \\ R & (S - R^{-1})^{-1} S R \end{bmatrix}, \quad \text{and} \quad M^{-1} = \begin{bmatrix} I - M_{12}^T N_{12} & N_{12}^T \\ N_{12} & J \end{bmatrix} > 0,$$

where $N_{12} = -JM_{12}$. Then

$$\dot{P}(\theta) = \begin{bmatrix} \dot{S} & -(\dot{S} + R^{-1}\dot{R}R^{-1}) \\ -(\dot{S} + R^{-1}\dot{R}R^{-1}) & \dot{S} + R^{-1}\dot{R}R^{-1} \end{bmatrix}.$$

Since conditions (27) are feasible with L, J, R_j, S_j, $j = 1, \ldots, m$, it can be checked using the Elimination Lemma [4, 32] that

$$X(\theta, \dot{\theta}) + U^T \Omega(\theta) V + V^T \Omega(\theta)^T U < 0, \tag{29}$$

must be feasible for some $\Omega(\theta)$, where

$$X(\theta, \dot{\theta}) = \begin{bmatrix} A_0^T P(\theta) + P(\theta) A_0 + \dot{P}(\theta) & P(\theta) B_0 & C_0^T \\ B_0^T P(\theta) & -M & D_0^T \\ C_0 & D_0 & -M^{-1} \end{bmatrix},$$

$$U = \begin{bmatrix} \mathcal{B}^T P(\theta) & 0 & \mathcal{D}_{pu}^T \end{bmatrix}, \quad V = \begin{bmatrix} \mathcal{C} & \mathcal{D}_{yq} & 0 \end{bmatrix},$$

$$A_0 = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & B_q(\theta) \\ 0 & 0 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 0 & 0 \\ C_p & 0 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0 & 0 \\ 0 & D_{pq}(\theta) \end{bmatrix}, \tag{30}$$

and $\mathcal{B}$, $\mathcal{D}_{pu}$, $\mathcal{C}$, and $\mathcal{D}_{yq}$ are given in (21).

Step 2. Design the gain-scheduled controller.

Solve the LMI feasibility problem (29) for $\Omega(\theta)$, which comprises the state-space matrices of the gain-scheduled controller that stabilizes the system. Moreover, $V(x, \theta) = x^T P(\theta) x$ is a parameter-dependent Lyapunov function that guarantees the stability of the closed loop system.

Note that in the above algorithm, since $P(\theta)$ is not an affine function of θ, solving (29) for a stabilizing gain-scheduled controller is not as easy as designing a gain-scheduled quadratic stabilizing controller in Section 3.1 (where $\Omega(\theta)$ is simply a convex combination of the controllers Ω_i that each correspond to a vertex of $\mathbf{\Delta}$). In order to construct the gain-scheduled stabilizing controller using the parameter-dependent Lyapunov functions introduced in Theorem 3.2, we need to solve the LMI feasibility problem (29) in real time. One approach is to solve LMI feasibility problem (29) numerically using an LMI solver [35]; another approach, which requires less on line computation time, is to find a feasible solution of (29) analytically (see [32]).

4 Numerical Examples

In Section 2 and Section 3, we observed that our approach is in general less conservative than structured scaling methods. We now illustrate this point through numerical examples.

4.1 Improved robustness analysis

The objective of the first example is to demonstrate that our approach yields significantly better results for robust stability analysis, as compared to the structured constant scaling methods. Consider a second order differential equation with parametric uncertainties

$$\ddot{x} + (1 - r(t)\cos\phi + r(t)\sin\phi + 0.5 r(t)^2 \sin 2\phi)\dot{x} + x = 0, \tag{31}$$

where $r(t)$ is a bounded uncertain time-varying parameter and ϕ is an (uncertain) angle lying in the sector $[0, \pi/4]$. Note that the parameters of (31) are not real-rational functions of the uncertainties. However, with a simple change of variables $\theta_1(t) = r(t)\cos\phi$ and $\theta_2(t) = r(t)\sin\phi$, we obtain the following LFR of (31) with $\Delta(t) = \mathbf{diag}(\theta_1(t), \theta_2(t))$:

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} q \\ p = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} q, &\quad q = \begin{bmatrix} \theta_1(t) & \\ & \theta_2(t) \end{bmatrix} p. \end{aligned} \tag{32}$$

This LFR is always well-posed. Define

$$\mathbf{\Theta} = \{(r(t)\cos\phi, r(t)\sin\phi) \mid \phi \in [0, \pi/4] \cup [\pi, 5\pi/4]\}.$$

This set is shown shaded in Fig. 4. Note that $\mathbf{\Theta}$ is not a polytope; we therefore use its polytopic covers in order to apply the results of Sections 2 and 3. The stability margin $\sigma_{\rm m}$ is the largest σ such that the stability of the system (32) can be guaranteed for any $|r(t)| \in [0, \sigma]$. We will compare the following robust stability analysis methods, using as the basis for comparison the robust stability margin that they can guarantee for system (32):

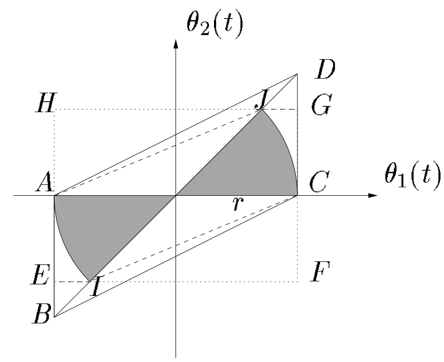


Figure 4: Uncertainty set $(\theta_1(t),\ \theta_2(t))$ and its polytopic cover.

1. *Hypercube cover, diagonal scaling [31, 4, 40].* This is equivalent to covering $\mathbf{\Theta}$ by the rectangle $EFGHE$ (dotted line).

2. *Polytopic cover, unstructured scaling (Corollary 2.2)* This can be interpreted as covering $\mathbf{\Theta}$ by the polytope $ABCDA$ (solid line).

3. *More accurate polytopic cover, unstructured scaling (Corollary 2.2).* This can be interpreted as covering $\mathbf{\Theta}$ by the polytope $AEICGJA$ (dashed line).

4. *Vertex scaling (Corollary 2.3).* Note that the LFR degree of system (32) is one, and therefore Corollary 2.3 can be applied with both polytopic covers $ABCDA$ and $AEICGJA$ for $\mathbf{\Theta}$.

For this example, the exact stability margin turns out to be 1. The lower bounds on the robust stability margin computed[3] using the approaches described above are given in Table 1. The results illustrate the observation that for linear systems affected by time-varying parameters, the approach described in this paper offers significant improvement for robustness analysis over the traditional structured scaling methods.

We note that when stability condition (10) with unstructured time-invariant multipliers or full block scalings is used, the resulting lower bound on the stability margin for case 3 with polytopic cover $AEICGJA$ is 0.9999. With this approach, however, we need to solve six LMIs for the six vertices of the polytopic cover, which requires approximately twice the computation as with constant unstructured scaling (Corollary 2.2) or with vertex scaling (Corollary 2.3).

[3]The actual calculation was performed by reformulating the lower bound calculation problem as a GEVP; see [41].

Polytope	No. of LMIs in (7)	Constant scaling	Vertex scaling
$EFGHE$	1	0.4874	—
$ABCDA$	2	0.8082	1.0000
$AEICGJA$	3	0.9024	1.0000

Table 1: A comparison of the stability analysis by using structured scaling and unstructured scalings.

4.2 Improved gain-scheduled controller synthesis

We have seen in Section 4.1 that our approach offers significant improvement over conventional structured scaling techniques for robustness analysis. We now demonstrate that similar improvements accrue with gain-scheduled controller synthesis as well.

Consider the system in Example 1 with an additional control input u and a measured output y, i.e.,

$$\ddot{x} + (1 - r(t)\cos\phi + r(t)\sin\phi + 0.5r(t)^2 \sin 2\phi)\dot{x} + x + u = 0, \quad y = x - \dot{x}. \tag{33}$$

Table 2 shows a comparison of the performance of different synthesis methods, based on the robust stabilizability margin that they can guarantee using quadratic Lyapunov functions for system (33). It can be seen that our approach using unstructured scalings yields significantly improved stabilizability margin than the design using structured scalings.

Type of scaling	Polytope	No. of LMIs in (23a)	Constant scaling
Diagonal	$EFGHE$	2	0.9985
Unstructured	$ABCDA$	4	1.3646
Unstructured	$AEICGJA$	6	1.8609

Table 2: A comparison of gain-scheduled controller synthesis by using structured scaling and unstructured scalings.

4.3 Improved stability analysis using parameter-dependent Lyapunov functions

In this example, we demonstrate the improvement achievable in robust stability analysis when parameter-dependent Lyapunov functions are employed (Corollary 2.4), as compared with quadratic Lyapunov functions (Corollary 2.2). We also compare the analysis results using an LFR model and a polytopic model to represent the uncertain system.

Consider the following parametric uncertain system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \left(\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} -24 & -3 \\ -24 & -3 \end{bmatrix} r(t)\cos\phi + \begin{bmatrix} 42 & 14 \\ 60 & 20 \end{bmatrix} r(t)\sin\phi + \begin{bmatrix} 28 & 3.5 \\ 40 & 5 \end{bmatrix} r^2(t)\sin 2\phi \right) \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{34}$$

where ϕ is a constant unknown parameter that lies in the set $[0, \pi/4]$, and $r(t)$ is a bounded time-varying uncertain parameter and satisfies $r(t) \in [0, r]$. The rate of variation of $r(t)$ is also bounded by k, i.e., $|\dot{r}(t)| \leq k$.

By defining new variables $\theta_1(t) = r(t)\cos\phi$, $\theta_2(t) = r(t)\sin\phi$, system (34) can be represented in an LFR model (Fig. 1), where $\Delta = \mathbf{diag}(\theta_1(t), \theta_2(t))$ is the diagonal parametric uncertainty. For this LFR model, Fig. 5 shows lower bounds of r under which robust stability of the system can be guaranteed using the parameter-dependent Lyapunov function (40) and a single quadratic Lyapunov function. The results show that the information on the bounds on the rate of variation of the uncertainties can be used to improve robust stability analysis. Similar improvement can be achieved with gain-scheduled controller design as well, when parameter-dependent Lyapunov functions are employed. However, the improvement comes with the cost of a larger number of LMIs and optimization variables.

For system (34), we also compare the performance analysis result using other parameter-dependent Lyapunov function approaches for affine parametric uncertan systems. In order to represent the system (34) in a model of affine parametric uncertain system (polytopic model), we define new variables $\hat{\theta}_1(t) = r(t)\cos\phi$, $\hat{\theta}_2(t) = r(t)\sin\phi$ and $\hat{\theta}_3(t) = r^2(t)\sin 2\phi$. The uncertainty set $\mathbf{\Theta} = \{(\hat{\theta}_1, \hat{\theta}_2, \hat{\theta}_3)\}$ is covered by a polytope, i.e.,

$$\mathbf{\Theta} \subset \left\{ (u_1, u_2, u_3) \,\middle|\, u_1 \in [0, r], u_2 \in [0, r/\sqrt{2}], u_3 \in [0, r^2] \right\}. \tag{35}$$

Then, it can be checked that even for time invariant uncertainties, the robust stability margin r of the uncertain system in polytopic model of (35) is less than 0.0266. If we further restrict the Lyapunov function to be of special forms, for example an affine function of the uncertainty variables as in [24], the lower bound of the robust stability margin that can be guaranteed for the polytopic model of (35) is even lower. In comparison, Fig. 5 shows that a lower bound of the robust stability margin of the LFR model that can be guaranteed by using a single quadratic Lyapunov function is 0.0266. If we use parameter-dependent Lyapunov functions for analyzing the LFR model, the lower bound of the robust stability margin can be further improved.

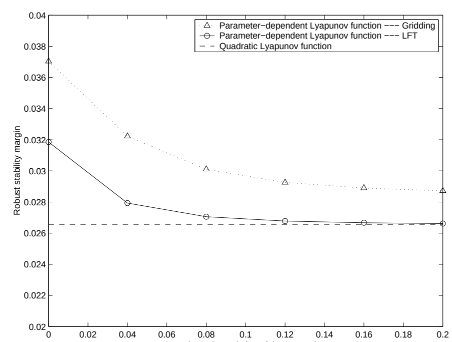


Figure 5: Robust stability analysis of system (34) with different bounds on the rate of variation of the uncertainty.

In this example, the analysis results with the polytopic model are conservative because the polytopic set coving the uncertainty set in (35) is conservative. If a polytopic model can be constructed to approximate the uncertain system more accurately, the analysis results with polytopic models can be improved. However, it is generally difficult to build an accurate polytopic cover for an uncertainty set involving multiple uncertainty variables[4]. Besides, an accurate polytopic cover generally includes many vertices, which may dramatically increase the number of LMIs.

Finally, we exercise the gridding method for the uncertain system (34) [18]. We choose a parameter-dependent Lyapunov function (40). The result of the gridding method provides an

[4]In [42], a procedure is proposed to construct a polytopic cover for uncertain systems whose uncertainty is included in a third order polynormial of a scaler uncertain variable.

upper bound for the analysis result using Corollary 2.4. However, the gridding method can not guarantee its analysis result, while the Corollary 2.4 gives a guaranteed lower bound of the robust stability margin. In addition, the gridding method requires solving more LMIs and the analysis result depends on choosing the gridding points.

4.4 Robust stability analysis of a nonlinear system

In this section, we demonstrate the application of the techniques presented in Section 2 towards the solution of stability analysis problems in nonlinear systems with real-rational nonlinearities. Consider an autonomous nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \dfrac{-2x_1 + x_2 + x_1x_2 + 2x_1^2 + 2.2x_2^2 - 5.2x_1^2x_2 - 2.2x_1x_2^2}{(1-x_1)(1-x_2)} \\ \dfrac{-10x_2 + 10x_1x_2 + 12.4x_2^2 - 2.4x_1^2x_2 - 12.4x_1x_2^2}{(1-x_1)(1-x_2)} \end{bmatrix}. \tag{36}$$

The system can be represented as an LFR with $\Delta = \mathbf{diag}(x_1, x_2)$ being the "uncertainty". We study the local stability around the origin, following the approach in [16]. Suppose that the quadratic Lyapunov function $V(\psi) = \psi^T Q^{-1} \psi$ guarantees the asymptotic stability of system (36) around the equilibrium point $x_i = 0$ when $|x_i| \leq \sigma,\ i = 1, 2$. Then, if Q satisfies $e_i^T Q e_i \leq \sigma^2$, $i = 1, 2$, it can be shown that $\mathcal{E}_{Q^{-1}} \triangleq \{x \mid x^T Q^{-1} x \leq 1\}$ is an invariant ellipsoid around the origin (see [16] for details). For fixed σ, we can then maximize the size of the invariant set $\mathcal{E}_{Q^{-1}}$, with "size" measured with different metrics. With $\sigma = 0.5$, we maximize the trace of Q, which has the interpretation of maximizing the sum of the squared semi-axis lengths of the invariant ellipsoid $\mathcal{E}_{Q^{-1}}$.

When the improved quadratic Lyapunov function search techniques that we presented in Section 2.1 are employed, much larger estimates for the region of stability of system (36) result. In Table 3, we list the sizes of the largest region of attraction that can be guaranteed using the conventional diagonal scalings, Corollary 2.2, and Corollary 2.3 respectively.

Quadratic stability criterion	Largest $\mathbf{Tr}(Q)$
Conventional method	0.1249
Corollary 2.2	0.4994
Corollary 2.3	0.4996

Table 3: Comparison of the largest invariant sets guaranteed by Theorem 2.1 and by the conventional diagonal scalings method.

4.5 Stabilizing nonlinear systems using gain-scheduled approach

In this example, we show that our approach can be applied to the design of output feedback controllers for nonlinear systems with real-rational nonlinearities.

Consider the following nonlinear system:

$$
\begin{aligned}
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} \dfrac{-2x_1 + x_2 + 72.9858x_1x_2 + 21.7683x_1^2 + 45.0738x_2^2 - 23.9491x_1^2x_2 - 15.3989x_1x_2^2}{(1-x_1)(1-x_2)} \\ \dfrac{-10x_2 + 90.6919x_1x_2 + 37.7998x_1^2 + 48.8071x_2^2 - 39.6367x_1^2x_2 - 43.8721x_1x_2^2}{(1-x_1)(1-x_2)} \end{bmatrix} \\
&\quad + \begin{bmatrix} 3.6443 \\ 5.5731 \end{bmatrix} u, \\
y &= 3.9401x_1 + 12.8006x_2.
\end{aligned}
$$

With the state variables (x_1, x_2) as "uncertain parameters", we design an output feedback controller such that:

1. $\mathcal{E}_P \triangleq \{x \mid x^T P x \leq 1\}$ is an invariant ellipsoid contained in the box $\mathcal{B}_\sigma \triangleq \{x \mid |x_i| \leq 0.4,\ i = 1, 2\}$, where P defines a quadratic Lyapunov function that guarantees local asymptotic stability around the origin.

2. For any initial condition that satisfies $|x_1(0)| \leq 0.16, |x_2(0)| \leq 0.16$, the peak value of the output

$$
y_{\text{peak}} \triangleq \max \{|y(t)| \mid t \geq 0,\ |x_1(0)| \leq 0.16,\ |x_2(0)| \leq 0.16\}
$$

is minimized. While there exist no methods that directly minimize y_{peak}, it is possible to minimize an upper bound $y_{\max}$ on y_{peak} via the optimization problem

$$
\texttt{minimize:} \quad y_{\max}^2, \quad \texttt{subject to:} \quad y_{\max}^2 \geq c_y R c_y^T, \tag{37}
$$

where R is the left upper block of P^{-1}, defined in (25); see [16] for details.

The gain-scheduling methods in [21, 16] that use structured scalings fail to produce a stabilizing controller; in other words y_{peak} is not even guaranteed to be finite with these methods. In contrast, the techniques presented in Sections 2 and 3 yield a gain-scheduled controller with an upper bound $y_{\max} = 4.3086$ for y_{peak}. This example illustrates the improvement achievable with our techniques for controller design for systems with real-rational nonlinearities.

4.6 A comparison of robust gain-scheduled control strategy with conventional gain-scheduled control strategy

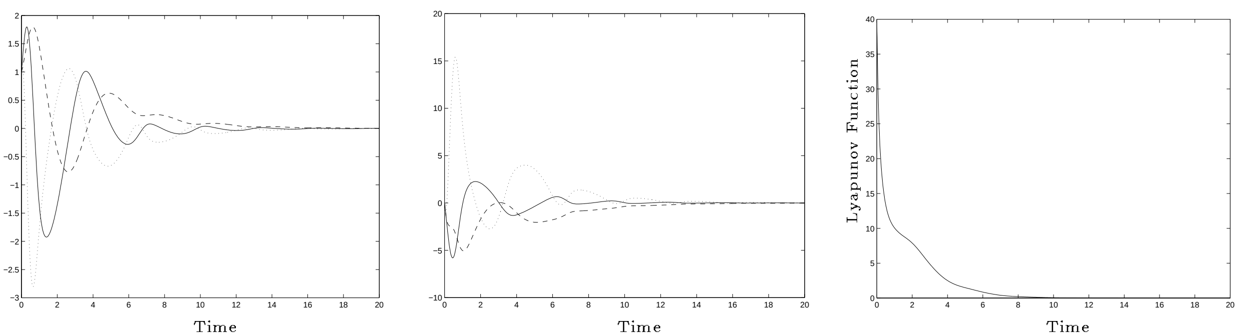
In the last example, we compare the robust gain-scheduled control strategy developed in this paper with a conventional gain-scheduled control strategy: Several controllers are designed for the system under different operating conditions, with the actual control law switching between the locally designed controllers under some scheduling scheme [15].

Consider the following parameter-dependent system from [15]:

$$
\begin{aligned}
\dot{x}(t) &= \begin{bmatrix} 0 & (2-\theta(t))^2 & 1 + 0.5\theta(t) + (2-\theta(t))^2 \\ 1 & 0 & 0.2 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t), \\
y(t) &= \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} x(t),
\end{aligned} \tag{38}
$$

where $\theta(t)$ is a measurable parametric uncertainty and satisfies $-1 \leq \theta(t) \leq 1$. This system can be represented as an LFR with $\Delta = \mathbf{diag}(\theta(t), \theta(t))$. For system (38), the conventional gain-scheduled control technique from [15] fails to internally stabilize the system. However the gain-scheduled controller obtained using the techniques of Section 3 does indeed stabilize the system. In fact, the solution of the GEVP (24) reveals that the system can be stabilized for $-1.8989 \leq \theta(t) \leq 1.8989$.

As an example, for the initial condition $x_1(0) = x_2(0) = x_3(0) = 1$ and with the uncertainty $\theta(t) = \cos(2t)$, Fig. 6 shows the state response of the closed-loop system with the gain-scheduled output feedback controller designed by applying the algorithm in Section 3.1.



(a) State response of the system $x_i(t), i = 1, 2, 3$.

(b) State response of the controller $x_{k_i}(t), i = 1, 2, 3$.

(c) Quadratic Lyapunov function.

Figure 6: Gain-scheduled control of uncertain system (38).

5 Conclusion

We have presented new algorithms for stability analysis and gain-scheduled controller synthesis for linear systems affected by parametric uncertainties. We have also established that these algorithms offer significant improvement over existing methods. The analysis and synthesis conditions are in the form of linear matrix inequalities; therefore, our algorithms can be very efficiently implemented numerically. The techniques presented in this paper can be applied to parameter-dependent nonlinear systems with real-rational nonlinearities. In addition, several of the techniques proposed in this paper can be extended to the solution of robust performance problems.

Appendix A

.1 Proof of Theorem 2.1

Suppose there exist $P = P^T > 0$, $G_\Delta \in \mathbf{C}^{d\times d}$ and $H_\Delta \in \mathbf{C}^{d\times d}$ satisfying (6) for all $\Delta \in \mathbf{\Delta}$. Consider system (4) for a $\Delta \in \mathbf{\Delta}$. The equations governing the system can be rewritten as

$$\dot{x} = Ax + (B_q\Delta)p, \quad p = C_px + (D_{pq}\Delta)p.$$

Now, for any $G_\Delta \in \mathbf{C}^{d\times d}$ and $H_\Delta \in \mathbf{C}^{d\times d}$, we have

$$x^T G_\Delta p = x^T G_\Delta C_p x + x^T G_\Delta (D_{pq}\Delta)p, \quad p^T H_\Delta p = p^T H_\Delta C_p x + p^T H_\Delta (D_{pq}\Delta)p,$$

or equivalently

$$\begin{bmatrix} x \\ p \end{bmatrix}^T \underbrace{\begin{bmatrix} G_\Delta C_p + C_p^T G_\Delta{}^* & G_\Delta(D_{pq}\Delta) - G_\Delta + C_p^T H_\Delta{}^* \\ (D_{pq}\Delta)^T G_\Delta{}^* - G_\Delta{}^* + H_\Delta C_p & H_\Delta(D_{pq}\Delta) + (D_{pq}\Delta)^T H_\Delta{}^* - H_\Delta - H_\Delta{}^* \end{bmatrix}}_{\mathcal{T}} \begin{bmatrix} x \\ p \end{bmatrix} = 0. \tag{39}$$

Thus system (4) can be described by the state equation, $\dot{x} = Ax + (B_q\Delta)p$ and the relation defined in (39). Then, we have

$$\begin{aligned} \frac{d}{dt}(x(t)^T P x(t)) &= \begin{bmatrix} x \\ p \end{bmatrix}^T \begin{bmatrix} PA + A^T P & P(B_q\Delta) \\ (B_q\Delta)^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} \\ &= \begin{bmatrix} x \\ p \end{bmatrix}^T \begin{bmatrix} PA + A^T P & P(B_q\Delta) \\ (B_q\Delta)^T P & 0 \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} x \\ p \end{bmatrix}^T \mathcal{T} \begin{bmatrix} x \\ p \end{bmatrix} < 0, \end{aligned}$$

with the last inequality following from (6).

.2 Proof of Corollary 2.2

Condition (6), with $G_\Delta = C_p^T M/2$ and $H_\Delta = ((D_{pq}\Delta)^T + I)M/2$, reduces to (7). The conclusion of the corollary then follows from Theorem 2.1.

.3 Proof of Corollary 2.3

Inequality (13) implies that there exists a quadratic Lyapunov function $V(\psi) = \psi^T P \psi$, such that $P > 0$ and $P(A+B_q\Delta_i(I-D_{pq}\Delta_i)^{-1}C_p)+(A+B_q\Delta_i(I-D_{pq}\Delta_i)^{-1}C_p)^T P < 0, \quad i = 1,\ldots,r$. (This follows from an argument exactly along the line of the proof of Theorem 2.1.) Now, since the LFR degree of system (3) is one, it turns out (see [6]) that $\mathbf{Co}\,\{A + B_q\Delta_i(I - D_{pq}\Delta_i)^{-1}C_p \mid i = 1,\ldots,r\} = \{A + B_q\Delta(I - D_{pq}\Delta)^{-1}C_p \mid \Delta \in \mathbf{Co}\{\Delta_1,\ldots,\Delta_r\}\}$. Therefore,

$$P(A + B_q\Delta(I - D_{pq}\Delta)^{-1}C_p) + (A + B_q\Delta(I - D_{pq}\Delta)^{-1}C_p)^T P < 0,$$

for any $\Delta \in \mathbf{Co}\{\Delta_1,\ldots,\Delta_r\}$, and system (3) is quadratically stable.

.4 Proof of Corollary 2.4

It is easy to establish using routine matrix algebra that if conditions (15) hold for all the vertices (Θ_i, Φ_i), they also hold for any uncertainty θ with $(\theta, \dot{\theta}) \in \mathbf{\Theta} \times \mathbf{\Phi}$. Following the same line as the proof of Theorem 2.1, it can be shown that the affinely parameter-dependent Lyapunov function

$$V(x) = x^T \left(Q_0 + \sum_{i=1}^{m} \theta_i Q_i \right)^{-1} x \tag{40}$$

provides a guarantee of the stability for system (4), where θ satisfies $(\theta, \dot{\theta}) \in \mathbf{\Theta} \times \mathbf{\Phi}$.

.5 Proof of Theorem 3.1

Since $\theta \in \gamma\mathbf{\Theta}$, we have $\Delta \in \gamma\mathbf{\Delta}$. Using Schur complements, observe that condition (22) is equivalent to

$$\begin{bmatrix} A_{\rm cl}^T(\theta)P + PA_{\rm cl}(\theta) & PB_{\rm cl}(\theta) & C_{\rm cl}^T(\theta) \\ B_{\rm cl}^T(\theta)P & -M & D_{\rm cl}^T(\theta) \\ C_{\rm cl}(\theta) & D_{\rm cl}(\theta) & -M^{-1} \end{bmatrix} < 0. \tag{41}$$

Then using (20), (21), and (30), inequality (41) can be written as

$$X(\theta) + U^T\Omega(\theta)V + V^T\Omega(\theta)^T U < 0, \tag{42}$$

where

$$X(\theta) = \begin{bmatrix} A_0^T P + PA_0 & PB_0 & C_0^T \\ B_0^T P & -M & D_0^T \\ C_0 & D_0 & -M^{-1} \end{bmatrix}, \quad U = \begin{bmatrix} \mathcal{B}^T P & 0 & \mathcal{D}_{pu}^T \end{bmatrix} \text{ and } V = \begin{bmatrix} \mathcal{C} & \mathcal{D}_{yq} & 0 \end{bmatrix}.$$

Partition P, P^{-1}, M and M^{-1} as

$$P = \begin{bmatrix} S & \star \\ \star & \star \end{bmatrix}, \quad P^{-1} = \begin{bmatrix} R & \star \\ \star & \star \end{bmatrix}, \quad M = \begin{bmatrix} \star & \star \\ \star & L \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} \star & \star \\ \star & J \end{bmatrix}.$$

Using this partitioning of P and M and the Elimination Lemma [4, 32] and the Completion Lemma [43], it is straight forward to verify that inequality (42) is feasible for some $\Omega(\Delta(\theta))$ if and only if

$$\begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} AR + RA^T & RC_p^T & (B_q\Delta)J \\ C_pR & -J & (D_{pq}\Delta)J \\ \hline J(B_q\Delta)^T & J(D_{pq}\Delta)^T & -J \end{array}\right] \begin{bmatrix} N_R & 0 \\ 0 & I \end{bmatrix} < 0,$$

$$\begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} A^TS + SA & S(B_q\Delta) & C_p^T L \\ (B_q\Delta)^T S & -L & (D_{pq}\Delta)^T L \\ \hline LC_p & L(D_{pq}\Delta) & -L \end{array}\right] \begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix} < 0, \tag{43a}$$

$$\begin{bmatrix} S & I \\ I & R \end{bmatrix} \geq 0, \quad \begin{bmatrix} L & I \\ I & J \end{bmatrix} \geq 0. \tag{43b}$$

Note that the left hand sides of (43a) are affine in Δ. Therefore conditions (43a) hold if and only if they hold for each vertex of $\gamma\mathbf{\Delta}$. Finally it is easily argued that if (43) is feasible with non-strict inequality, it is also feasible with strict inequality. Thus we get condition (23).

.6 Proof of Theorem 3.2

Condition $\begin{bmatrix} L & \Delta(\theta)X \\ X\Delta(\theta) & X \end{bmatrix} > 0$ implies $L > \Delta X \Delta$. Multiplying inequality (27b) on the left and right by $\mathbf{diag}(I, \Delta, I)$, we get

$$\begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix}^T \left[\begin{array}{cc|c} F_{11} & \hat{F}_{12} & F_{13} \\ \hat{F}_{12}^T & -L & \hat{F}_{23} \\ \hline F_{13}^T & \hat{F}_{23}^T & -L \end{array}\right] \begin{bmatrix} N_S & 0 \\ 0 & I \end{bmatrix} < 0, \tag{44}$$

where $\hat{F}_{12} = \left(S_0 + \sum_{j=1}^{m} \theta_j(t) S_j \right) B_q(\theta)$ and $\hat{F}_{23} = D_{pq}(\theta)^T L$. Therefore (27b) holds for all $(\theta, \dot{\theta}) \in \mathbf{\Theta} \times \mathbf{\Phi}$ if and only if (44) holds for all $(\theta, \dot{\theta}) \in \mathbf{\Theta} \times \mathbf{\Phi}$. Using a similar argument as the one in the proof of Theorem 3.1, it can be checked that

$$\begin{bmatrix} A_{\mathrm{cl}}(\theta)^T P(\theta) + P(\theta) A_{\mathrm{cl}}(\theta) + C_{\mathrm{cl}}(\theta)^T M C_{\mathrm{cl}}(\theta) + \dot{P}(\theta) & P(\theta) B_{\mathrm{cl}}(\theta) + C_{\mathrm{cl}}(\theta)^T M D_{\mathrm{cl}}(\theta) \\ B_{\mathrm{cl}}(\theta)^T P(\theta) + D_{\mathrm{cl}}(\theta)^T M C_{\mathrm{cl}}(\theta) & -M + D_{\mathrm{cl}}(\theta)^T M D_{\mathrm{cl}}(\theta) \end{bmatrix} < 0,$$

where $P(\theta)$ and M are defined in (28). Thus we conclude that $P(\theta)$ defines a parameter-dependent Lyapunov function that guarantees robust stability.

Appendix B

Theorem B.1 *System (4) is quadratically stable if and only if there exists $P = P^T > 0$ such that for every uncertainty $\Delta \in \mathbf{\Delta}$ defined in (5), the following condition holds*

$$\begin{bmatrix} A^T P + PA - C_p^T C_p & P(B_q \Delta) + C_p^T - C_p^T (D_{pq}\Delta) \\ (B_q\Delta)^T P + C_p - (D_{pq}\Delta)^T C_p & (D_{pq}\Delta) + (D_{pq}\Delta)^T - (D_{pq}\Delta)^T (D_{pq}\Delta) - I \end{bmatrix} < 0. \quad (45)$$

Moreover, if $D_{pq} = 0$, condition (45) is equivalent to (11) with $S_i = -Q = I$.

Proof: The connection between (45) and (11) is straightforward when $D_{pq} = 0$. Therefore, we only need to show that condition (45) holds for the more general case.

Sufficiency

Condition (45), from Schur complements lemma, is equivalent to the condition that for all $\Delta \in \mathbf{\Delta}$

$$\begin{aligned} P(A + B_q \Delta (I - D_{pq}\Delta)^{-1} C_p) + (A + B_q \Delta (I - D_{pq}\Delta)^{-1} C_p)^T P \\ + P B_q \Delta (I - D_{pq}\Delta)^{-1} (I - D_{pq}\Delta)^{-T} \Delta^T B_q^T P < 0. \end{aligned}$$

Quadratic stability of system (4) follows immediately. Also note that condition (45) implies that system (4) is well-posed.

Necessity

Suppose that system (4) is quadratically stable. Then, there exists $P = P^T > 0$ such that

$$L(\Delta) = P(A + B_q \Delta (I - D_{pq}\Delta)^{-1} C_p) + (A + B_q \Delta (I - D_{pq}\Delta)^{-1} C_p)^T P < 0, \quad \forall \Delta \in \mathbf{\Delta}. \quad (46)$$

Let $\eta = \sup_{\Delta \in \mathbf{\Delta}} (\lambda_{\max}(L(\Delta)))$, where $\lambda_{\max}$ denotes the largest eigenvalue. Obviously $\eta \le 0$. We show that $\eta < 0$. Suppose otherwise, i.e., $\eta = 0$. Since $\mathbf{\Delta}$ is a compact set and $\lambda_{\max}(\cdot)$ is a continuous function, there must then exist some $\Delta \in \mathbf{\Delta}$ such that $\lambda_{\max}(L(\Delta)) = \eta = 0$, which contradicts the assumption (46).

Since the system is well-posed (otherwise, it cannot be quadratically stable), $\det(I - D_{pq}\Delta) \neq 0$ for all $\Delta \in \mathbf{\Delta}$. This implies $(I - D_{pq}\Delta)(I - D_{pq}\Delta)^T > 0$ for all $\Delta \in \mathbf{\Delta}$. Following the same argument as before, we obtain $\inf_{\Delta \in \mathbf{\Delta}} \lambda_{\min}((I - D_{pq}\Delta)(I - D_{pq}\Delta)^T) = \tau > 0$, or equivalently $\|(I - D_{pq}\Delta)^{-T}\|^2 \le 1/\tau$ for all $\Delta \in \mathbf{\Delta}$. Define $\tilde{P} = \xi P$ where

$$0 < \xi < -\frac{\tau \eta}{\lambda_{\max}(P B_q B_q^T P)\mu}, \quad \|\Delta\|^2 \le \mu, \quad \mu > 0.$$

It is easy to check that $\tilde{P}$ is a feasible solution for inequality (45). Thus we have shown that (45) is a necessary condition for quadratic stability. $\square$

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