Algorithms and Software for LMI Problems in Control*

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A number of important problems from system and control theory can be numerically solved by reformulating them as convex optimization problems with linear matrix inequality (LMI) constraints. While numerous articles have appeared cataloging applications of LMIs to control system analysis and design, there have been few publications in the control literature describing the numerical solution of these optimization problems. The purpose of this article is to provide an overview of the state of the art of numerical algorithms for LMI problems, and of the available software.

Introduction

A wide variety of problems in systems and control theory can be cast or recast as *semidefinite programming* (SDP) problems[1], that is, problems of the form

$$\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & C + \sum_{i=1}^{m} y_i A_i \leq 0, \end{array} \tag{1}$$

where $y \in \mathbf{R}^m$ is the variable and the matrices $C = C^T \in \mathbf{R}^{n \times n}$, and $A_i = A_i^T \in \mathbf{R}^{n \times n}$ are given. The inequality sign denotes matrix inequality, *i.e.*, the matrix $C + \sum_i y_i A_i$ is negative semidefinite. The constraint

$$C + \sum_{i=1}^{m} y_i A_i \leq 0$$

is called a *linear matrix inequality* (LMI). In other words, SDPs are convex optimization problems with a linear objective function and linear matrix inequality (LMI) constraints.

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[1] We shall use SDP to mean both "semidefinite programming", as well as a "semidefinite program", i.e., a semidefinite programming problem.

Though the form of the SDP (1) appears very specialized, it turns out that it is widely encountered in systems and control theory. Examples include: multicriterion LQG, synthesis of linear state feedback for multiple or nonlinear plants ("multi-model control"), optimal state-space realizations of transfer matrices, norm scaling, synthesis of multipliers for Popov-like analysis of systems with unknown gains, robustness analysis and robust controller design, gain-scheduled controller design, and many others.

For a few very special cases there are "analytical solutions" to SDPs (via Riccati equations for the ones encountered with $\mathbf{H}_2$ and $\mathbf{H}_\infty$ control [2], for example), but in general they can be solved numerically very efficiently. In many cases—for example, with multi-model control [3]—the LMIs encountered in SDPs in systems and control theory have the form of simultaneous (coupled) Lyapunov or algebraic Riccati inequalities; using recent interior-point methods such problems can be solved in a time that is roughly comparable to the time required to solve the same number of (uncoupled) Lyapunov or Algebraic Riccati equations [3, 4]. Therefore the computational cost of extending current control theory that is based on the solution of algebraic Riccati equations to a theory based on the solution of (multiple, simultaneous) Lyapunov or Riccati inequalities is modest.

A number of publications can be found in the control literature that survey applications of SDP to the solution of system and control problems. Perhaps the most comprehensive list can be found in the book [3]. Since its publication, a number of papers have appeared chronicling further applications of SDP in control; we cite for instance the survey article [5] that appeared in this magazine, and the special issue of the International Journal of Robust and Nonlinear Control on *Linear Matrix Inequalities in Control Theory and Applications*, published recently, in November-December, 1996 [6]. The growing popularity of LMI methods for control is also evidenced by the large number of publications in recent control conferences.

Special classes of the SDP have a long history in optimization as well. For example, certain eigenvalue minimization problems that can be cast as SDPs have been used for obtaining bounds and heuristic solutions for combinatorial optimization problems (see [7, 8] and [9, Chapter 9]). The efficiency of recent interior-point methods for SDP, which is directly responsible for the popularity of SDP in control, has therefore also attracted a great deal of interest in optimization circles, overshadowing earlier solution methods based on techniques from nondifferentiable optimization [8, 10, 11, 12, 13]. At every major optimization conference, there are workshops and special sessions devoted exclusively to SDP, and a special issue of *Mathematical Programming* has recently been devoted to SDP [14]. This interest was primarily motivated by applications of SDP in combinatorial optimization but, more recently, also by the applications in control.

The primary purpose of this article is to provide an overview of the state of the art of numerical algorithms for LMI problems, and of the available software. We first review the definition and some basic properties of the semidefinite programming problem. We then describe recent developments in interior-point algorithms and available software. We conclude with some extensions of SDP.

Semidefinite programming

In this section we provide a brief introduction to the semidefinite programming problem. For more extensive surveys on the theory and applications of SDP, we refer to Alizadeh [15], Boyd *et al.* [3], Lewis and Overton [16], Nesterov and Nemirovskii [17, §6.4], and Vandenberghe and Boyd [18].

We have already defined an SDP formally in (1). To distinguish it from other formulations, we will refer to (1) as an SDP in *inequality form*. The optimization problem

$$\begin{array}{ll} \text{maximize} & \mathbf{Tr} CX \\ \text{subject to} & X \geq 0 \\ & \mathbf{Tr} A_i X + b_i = 0, \quad i = 1, \ldots, m \end{array} \tag{2}$$

is called an SDP in *equality form*. Here, the variable is the matrix $X = X^T \in \mathbf{R}^{n \times n}$, and $\mathbf{Tr}$ stands for trace, ie., sum of the diagonal entries of a square matrix. The SDP (1) can be easily converted into (2) and vice-versa, so it is a matter of convention what we consider as the 'standard' form (although the inequality form appears to be more appropriate for control theory).

It turns out that the the semidefinite programs (1) and (2) can be regarded as generalizations of several important optimization problems. For example, the linear program (LP)

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & x \geq 0 \\ & a_i^T x + b_i = 0, \quad i = 1, \ldots, m, \end{array} \tag{3}$$

in which the inequality $x \geq 0$ denotes componentwise inequality, can be expressed as an SDP (2) with $A_i = \mathbf{diag}(a_i)$ and $C = \mathbf{diag}(c)$, and $X = \mathbf{diag}(x)$. Semidefinite programming can also be regarded as an extension of linear programming where the componentwise inequalities between vectors are replaced by matrix inequalities, or, equivalently, the first orthant is replaced by the cone of positive semidefinite matrices.

It can be shown that problems (1) and (2) are duals of each other. More precisely, if $\ell^\star$ is the optimal value of (2) and $u^\star$ is the optimal value of (1), then we have:

- *weak duality:* $u^\star \geq \ell^\star$;
- *strong duality:* If (1) is strictly feasible (*i.e.*, there exists a y with $C + \sum_i y_i A_i < 0$) or (2) is strictly feasible (*i.e.*, there exists an $X > 0$ with $\mathbf{Tr} A_i X + b_i = 0$), then $u^\star = \ell^\star$.

The result follows from standard convex optimization duality. (A stronger duality theory that does not require strict feasibility was recently developed by Ramana, Tunçel and Wolkowicz [19].) Some connections between SDP duality and duality in control are explored in [20].

If we assume that both (1) and (2) are strictly feasible, then the optimal values in both problems are attained, and the solutions are characterized by the optimality conditions

$$\begin{array}{l} X \geq 0, Z \geq 0 \\ \mathbf{Tr} A_i X + b_i = 0, \quad i = 1, \ldots, m \\ Z + C + \sum_{i=1}^m y_i A_i = 0 \\ XZ = 0. \end{array} \tag{4}$$

The first three conditions state feasibility of X, Z and y. The last condition is called *complementary slackness.*

Interior-point methods

Brief history

The ideas underlying interior-point methods for convex optimization can be traced back to the sixties; see *e.g.*, Fiacco and McCormick [21], Lieu and Huard [22], and Dikin [23]). Interest in them was revived in 1984, when Karmarkar introduced a polynomial-time interior-point method for LP [24]. In 1988 Nesterov and Nemirovskii [25] showed that those interior-point methods for linear programming can, in principle, be generalized to all convex optimization problems. The key element is the knowledge of a barrier function with a certain property called *self-concordance.* Linear matrix inequalities are an important class of convex constraints for which readily computable self-concordant barrier functions are known, and, therefore, interior-point methods are applicable.

Independently of Nesterov and Nemirovskii, Alizadeh [26] and Kamath and Karmarkar [27, 28] generalized interior-point methods from linear programming to semidefinite programming. Vast progress has been made in the last two years, and today almost all interior-point methods for linear programming have been extended to semidefinite programming. This recent research has largely concentrated on primal-dual methods in the hope of emulating the excellent performance of primal-dual interior-point methods for large-scale linear programming [29, 30]. The remainder of this section will concentrate on this recent work. We should mention however that other methods have been used successfully, *e.g.*, the ellipsoid algorithm, the method of alternating projections, and primal interior-point methods such as the projective algorithm and the method of centers. We refer to [5, p.80] or [3, §2] for surveys of these earlier methods.

Primal-dual methods for SDP

The most promising methods for semidefinite programming solve the two problems (1) and (2) simultaneously. These primal-dual methods are usually interpreted as methods for following the *primal-dual central path*, which is defined as the set of solutions $X(\mu)$, $Z(\mu)$, $y(\mu)$ of the nonlinear equations

$$\begin{array}{l} X \geq 0, Z \geq 0 \\ \mathbf{Tr} A_i X + b_i = 0, \quad i = 1, \ldots, m \\ Z + C + \sum_{i=1}^m y_i A_i = 0 \\ XZ = \mu I, \end{array} \tag{5}$$

where $\mu \geq 0$ is a parameter. Note that these conditions are very similar to the optimality conditions (4). The only difference is the last condition, $XZ = \mu I$, which replaces the complementary slackness condition $XZ = 0$. It can be shown that the solution of (5) is unique for $\mu > 0$ (assuming strict primal and dual feasibility), and that $X(\mu)$, $Z(\mu)$, $y(\mu)$ approach optimality if μ goes to zero.

The central points $X(\mu)$, $y(\mu)$, $Z(\mu)$ are also the minimizers of two convex functions: $X(\mu)$ minimizes

$$\varphi_p(\mu, X) = -\mathbf{Tr} CX - \mu \log\det X^{-1}$$

over all $X > 0$ with $\mathbf{Tr} A_i X + b_i = 0$; $y(\mu)$ minimizes

$$\varphi_d(\mu, y) = b^T y - \mu \log\det \left(-C - \sum_{i=1}^m y_i A_i \right),$$

over all y with $C + \sum_{i=1}^m y_i A_i < 0$. These two functions are not only convex, but also self-concordant, which allows us to apply Nesterov and Nemirovskii's theory for proving polynomial complexity of interior-point methods.

The idea behind most interior-point methods is to generate a sequence of X, y, Z that converge to optimality, by following the central path for decreasing values of μ. We will not discuss in detail how this is done in practice, but instead concentrate on the most expensive step of each iteration: the computation of primal and dual search directions δX, δy and δZ. These search directions can usually be interpreted as Newton directions for solving the set of nonlinear equations (5), *i.e.*, we compute the search directions by linearizing (5) around the current iterates and solving a set of linear equations. Several possibilities exist to linearize these equations, and the particular choice distinguishes the different interior-point methods, as we will now explain.

Let $X > 0$, $Z > 0$, y be the current iterate. For simplicity we assume that these points are feasible, *i.e.*, we assume $\mathbf{Tr} A_i X + b_i = 0$ and $Z + C + \sum_{y=1}^m y_i A_i = 0$, although the methods are readily extended to infeasible starting points. A first possibility to linearize $XZ = \mu I$ is to write it as $X = \mu Z^{-1}$, and to linearize the equations (5) as

$$\begin{aligned} \mathbf{Tr} A_i \delta X &= 0, \quad i = 1, \ldots, m \\ \delta Z + \sum_{i=1}^m \delta y_i A_i &= 0 \\ \delta X + \mu Z^{-1} \delta Z Z^{-1} &= \mu Z^{-1} - X. \end{aligned}$$

If we eliminate δZ from the second and third equations, we obtain

$$\mathbf{Tr} A_i \delta X = 0, \quad i = 1, \ldots, m \tag{6}$$

$$-\mu^{-1} Z \delta X Z + \sum_{i=1}^m \delta y_i A_i = \mu^{-1} Z X Z - Z. \tag{7}$$

This is a set of $m + n(n+1)/2$ equations in the $m + n(n+1)/2$ variables δy, $\delta X = \delta X^T \in \mathbf{R}^{n \times n}$.

In the special case of the LP (3), where all matrices are diagonal, we can write $\delta Z = \mathbf{diag}(\delta z)$, $\delta X = \mathbf{diag}(\delta x)$, and

$$\left[\begin{array}{cc} A^T & 0 \\ -\mu^{-1} Z^2 & A \end{array} \right] \left[\begin{array}{c} \delta x \\ \delta y \end{array} \right] = \left[\begin{array}{c} 0 \\ \mu^{-1} Z^2 x - z \end{array} \right].$$

where $A = [a_1 \ \ldots \ a_m]$.

A second possibility for linearizing the equation $XZ = \mu I$ is to write it as $Z = \mu X^{-1}$, which leads to

$$\begin{aligned} \mathbf{Tr} A_i \delta X &= 0, \quad i = 1, \ldots, m \\ \delta Z + \sum_{i=1}^{m} \delta y_i A_i &= 0 \\ \mu X^{-1} \delta X X^{-1} + \delta Z &= -Z + \mu X^{-1}. \end{aligned}$$

Eliminating δZ, we obtain

$$\begin{aligned} \mathbf{Tr} A_i \delta X &= 0, \quad i = 1, \ldots, m \\ -\mu X^{-1} \delta X X^{-1} + \sum_{i=1}^{m} \delta y_i A_i &= Z - \mu X^{-1}. \end{aligned}$$

Specialized to linear programming the equations become

$$\begin{bmatrix} A^T & 0 \\ -\mu X^{-2} & A \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} 0 \\ z - \mu X^{-1} e \end{bmatrix}.$$

The first SDP methods were based on these primal or dual scalings (see for example, Nesterov and Nemirovskii [17], Alizadeh [26], and Vandenberghe and Boyd [4]). In linear programming, however, the primal and dual scalings are rarely used in practice. Instead, one usually prefers a primal-dual symmetric scaling introduced by Kojima, Mizuno and Yoshise [31]. For linear programming the resulting equations for the search directions have the form

$$\begin{bmatrix} A^T & 0 \\ -X^{-1} Z & A \end{bmatrix} \begin{bmatrix} \delta x \\ \delta y \end{bmatrix} = \begin{bmatrix} 0 \\ z - \mu X^{-1} e. \end{bmatrix}. \tag{8}$$

These equations are obtained by linearizing $XZ = \mu I$ as

$$X \delta Z + \delta X Z = \mu I - XZ. \tag{9}$$

Several researchers have demonstrated that methods that use this primal-dual symmetric scaling can achieve a higher accuracy than methods based on the the primal or dual scaling (see for example Wright [32]), and therefore the symmetric scaling is the basis of all practical LP interior-point methods.

The extension of this symmetric primal-dual scaling to SDP is not straightforward: The linearization (9) leads to a linear system

$$\mathbf{Tr} A_i \delta X = 0, \; i = 1, \ldots, m \tag{10}$$

$$-X^{-1} \delta X Z + \sum_{i=1}^{m} \delta y_i A_i = Z - \mu X^{-1} \tag{11}$$

but unfortunately the solution δX is not symmetric in general. Much of the most recent research in SDP has therefore concentrated on extending the primal-dual symmetric scaling from LP to SDP, and, as a result of this effort, very rapid progress has been made in the last two years. Among the proposed symmetric primal-dual algorithms, three variations seem to be the most promising. Helmberg, Rendl, Vanderbei, and Wolkowicz [33], Kojima, Shidoh and Hara [34], and Monteiro [35] solve (10) and (11) and linearize the resulting δX. Alizadeh, Haeberly and Overton [36] first write $XZ = \mu I$ as $XZ + ZX = 2\mu I$ and then linearize this as

$$X\delta Z + \delta X Z + Z\delta X + \delta Z X = 2\mu I - XZ - ZX.$$

The resulting δX and δZ are automatically symmetric. Finally, Nesterov and Todd [37, 38], and recently Sturm and Zhang [39], have defined a third direction, obtained as follows. First a matrix R is computed such that $R^T X R = \Lambda^{1/2}$ and $R^T Z^{-1} R = \Lambda^{1/2}$, where Λ is a diagonal matrix with as diagonal elements the eigenvalues of XZ. One then solves the equations

$$\mathbf{Tr} A_i \delta X = 0, \; i = 1, \ldots, m \tag{12}$$

$$-RR^T \delta X RR^T + \sum_{i=1}^{m} \delta y_i A_i = Z - \mu X^{-1}. \tag{13}$$

to obtain the search directions δX, δZ, δy. Numerical details on this method can be found in Todd, Toh and Tütüncü [40]. Finally, Kojima, Shindoh and Hara [34], Monteiro [35], and Monteiro and Zhang [41] have presented unifying frameworks for primal-dual methods.

Some other important recent articles and reports are listed in the references of this paper[2].

Software packages

Several researchers have made available software for semidefinite programming. The first implementation of an interior-point method for SDP was by Nesterov and Nemirovskii in [65], using the projective algorithm [17]. Matlab's LMI Control Toolbox [66] is based on the same algorithm, and offers a graphical user interface and extensive support for control applications. The code SP [67] is based on a primal-dual potential reduction method with the Nesterov and Todd scaling. The code is written in C with calls to BLAS and LAPACK and includes an interface to Matlab. SDPSOL [68] and LMITOOL [69] offer user-friendly interfaces to SP that simplify the specification of SDPs where the variables have matrix structure. The Induced-Norm Control Toolbox [70] is a toolbox for robust and optimal control, in turn based on LMITOOL.

Several implementations of the most recent primal-dual methods are also available now. SDPA [71] is a C++ code, based on the algorithm of Kojima, Shindoh and Hara [34]. CSDP [72] is a C implementation of the algorithm of Helmberg, Rendl, Vanderbei, and Wolkowicz [33]. SDPHA [73] is a Matlab implementation of a homogeneous formulation of the different primal-dual methods described above. SDPT3 [74] is a Matlab implementation of the most important infeasible primal-dual path-following methods. SDPPACK [75] is an implementation of the algorithm of [36]. It is written in Matlab, with critical parts written in C to increase the efficiency. It also provides the useful feature of handling quadratic and linear constraints directly.

[2] Most recent papers are available at the semidefinite programming homepage maintained by Cristoph Helmberg (`http://www.zib-berlin.de/~bzfhelmb/semidef.html`) and the interior-point archive at Argonne National Laboratory (`http://www.mcs.anl.gov/home/otc/InteriorPoint/index.html`).

Extensions

The determinant maximization problem

In their survey of LMI problems in control, Boyd *et al.* [3] also considered an extension of the SDP (1), which was discussed in more detail in [76]. This extension can be written in the following form:

$$\begin{array}{ll} \text{minimize} & b^T y - \log\det\left(-D - \sum_{i=1}^m y_i B_i\right) \\ \text{subject to} & C + \sum_{i=1}^m y_i A_i \leq 0 \\ & D + \sum_{i=1}^m y_i B_i < 0. \end{array} \tag{14}$$

We will call this problem a maxdet-problem[3], since in most cases of interest (*e.g.*, ellipsoidal approximation problems) the term $b^T y$ is absent, so the problem reduces to maximizing the determinant of $-D - \sum_i y_i B_i$ subject to an additional LMI constraint. The maxdet-problem is a convex optimization problem since the function $\log\det X^{-1}$ is convex on the positive definite cone.

An equivalent problem is the maxdet-problem in equality form:

$$\begin{array}{ll} \text{maximize} & \mathbf{Tr} CX + \mathbf{Tr} DW + \log\det W \\ \text{subject to} & X \geq 0, \;\; W > 0 \\ & \mathbf{Tr} A_i X + \mathbf{Tr} B_i W + b_i = 0, \; i = 1, \ldots, m, \end{array} \tag{15}$$

where $X = X^T$ and $W = W^T$ are the variables. Again it can be shown that problems (15) and (14) are duals.

Maxdet-problems arise in many fields, including computational geometry, statistics, and information and communication theory. Therefore the theory behind their solution is of great interest, and the resulting algorithms have wide application. A list of applications and an interior-point method for the maxdet-problem are described in [76]. Software for solving maxdet-problems is available in [77], and has been incorporated in SDPSOL [68].

The generalized eigenvalue minimization problem

A third standard problem from [3] is the generalized eigenvalue minimization problem. Suppose we have a pair of matrices $(A(x), B(x))$, both affine functions of x. In order to minimize their maximum generalized eigenvalue, we can solve the optimization problem

$$\begin{array}{ll} \text{minimize} & t \\ \text{subject to} & tB(x) - A(x) \geq 0 \\ & B(x) \geq 0. \end{array} \tag{16}$$

[3]The problem was denoted CP in [3].

This is called a generalized linear-fractional problem. It includes the linear fractional problem

$$\begin{array}{ll} \text{minimize} & \dfrac{c^T x + d}{e^T x + f} \\ \text{subject to} & Ax + b \geq 0, \quad e^T x + f > 0 \end{array}$$

as a special case.

Problem (16) is not a semidefinite program, however, because of the bilinear term $tB(x)$. It is a quasi-convex problem, and can still be efficiently solved. See Boyd and El Ghaoui [78], Haeberly and Overton [79], and Nesterov and Nemirovskii [17, 80, 81] for details on specialized algorithms, and [3] for applications of this problem in control. An implementation of the Nesterov and Nemirovskii algorithm is also provided in the LMI Control toolbox [66].

The bilinear matrix inequality problem

We finally consider an extension of the SDP (1), obtained by replacing the linear matrix inequality constraints by a *quadratic* matrix inequality,

$$\begin{array}{ll} \text{minimize} & b^T y \\ \text{subject to} & C + \sum_{i=1}^{m} y_i A_i + \sum_{i,j=1}^{m} y_i y_j B_{ij} \leq 0. \end{array} \tag{17}$$

This problem is nonconvex, but it is extremely general. For example, if the matrices C, A_i, and B_{ij} are diagonal, the constraint in (17) reduces to a set of n (possibly indefinite) quadratic constraints in x. Problem (17) therefore includes all quadratic optimization problems. It also includes all polynomial problems (since by introducing new variables, one can reduce any polynomial inequality to a set of quadratic inequalities), all $\{0, 1\}$ and integer programs, etc.

In control theory, a more restricted *bilinear* form seems to be general enough. Here we split the variables in two vectors x and y, and replace the constraint by a *bilinear* (or bi-affine) matrix inequality (BMI):

$$\begin{array}{ll} \text{minimize} & c^T x + b^T y \\ \text{subject to} & D + \sum_{i=1}^{m} y_i A_i + \sum_{k=1}^{l} x_k B_k + \sum_{i=1}^{m} \sum_{k=1}^{l} x_i y_k C_{ik} \leq 0. \end{array} \tag{18}$$

The problem data are the vectors $c \in \mathbf{R}^m$ and $b \in \mathbf{R}^l$ and the symmetric matrices A_i, B_k, and $C_{ik} \in \mathbf{R}^{n \times n}$.

BMIs include a wide variety of control problems, including synthesis with structured uncertainty, fixed-order controller design, decentralized controller synthesis etc. (see Safonov, Goh, and others [82, 83, 84, 85, 86, 87], El Ghaoui and Balakrishnan [88], etc). The fundamental difference with LMIs is that BMI problems are non-convex, and no non-exponential-time algorithms for their solution are known to exist. The algorithms described in the above references are either local methods that alternate between minimizing over x and y, or global (branch and bound) techniques based on the solution of a sequence of LMI problems.

Conclusion

The current state of research on LMIs in control can be summarized:

- There has been intensive research on identifying control problems that can be cast in terms of LMIs, and those for which an LMI formulation is unlikely to exist. In the latter case, bilinear matrix inequalities (BMIs) have been recognized as a useful formulation.
- The combined activity in mathematical programming and control theory has led to very rapid progress in interior-point algorithms for solving SDPs, focusing on local convergence rates, worst-case complexity, etc., and on extending to SDP the sophisticated and efficient primal-dual interior-point methods developed for linear programming.
- Several basic software implementations of interior-point methods for SDP have become available. These codes have proven useful for small to medium-sized problems.

Thus LMIs are becoming basic tools in control, much the way Riccati equations became basic tools in the 1960s. At the same time, the current strong interest in the mathematical programming community is leading to more powerful algorithms for the LMI and BMI problems that arise in control. We expect that this research will lead to a second generation of general-purpose LMI codes, which will exploit more problem structure (*e.g.*, sparsity) to increase the efficiency. The analogy with linear programming illustrates the ramifications. To a large extent, linear programming owes its success to the existence of general-purpose software for large sparse LPs. On a more modest scale, the availability of efficient general-purpose software for SDP would have a similar effect: it would make it possible to routinely solve large SDPs in a wide variety of applications.

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