H^∞ Controller Synthesis with Time-Domain Constraints

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Abstract

For a class of discrete-time linear time-invariant (LTI) plants, we present necessary and sufficient conditions for the existence of a discrete-time LTI controller that ensures that the H^∞ norm of a closed-loop map of interest is less than a prespecified level, subject to time-domain constraints on this closed-loop map. In cases when such an LTI controller does not exist, we show that the problem of finding an LTI controller that satisfies the input-output constraints with the smallest error can be posed as a convex optimization problem based on linear matrix inequalities. We also consider briefly other related problems that can be solved completely using a similar approach, namely controller synthesis to minimize the optimally scaled H^∞ norm, and controller synthesis to maximize the guaranteed dissipation.

1 Introduction

A standard framework for controller synthesis is shown in Figure 1. P is a linear time-invariant (LTI) plant

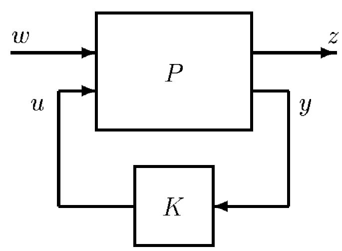

Figure 1: Controller design framework

and K is the controller. $w : \mathbf{Z}_+ \to \mathbf{R}_{n_w}$ is the exogenous input, $u : \mathbf{Z}_+ \to \mathbf{R}_{n_u}$ is the control input, $y : \mathbf{Z}_+ \to \mathbf{R}_{n_y}$ is the observed output, and $z : \mathbf{Z}_+ \to \mathbf{R}_{n_z}$ is the output of interest. In this paper, we will restrict our attention to LTI controllers K; thus the closed-loop system is an LTI system. We let H_{zw} denote the closed-loop transfer matrix from w to z.

Our objective in this paper, for the most part, is to design K so as to minimize the H^∞ norm of H_{zw}, subject to constraints on the *transient-response* at z due to some specified reference input at w (we will refer to such constraints as "transient-response tracking" constraints). In cases when a transient-response error can be tolerated, we study the tradeoff between the error in transient-response tracking and the H^∞ norm of H_{zw}. We also consider briefly other related problems, namely: (i) LTI controller synthesis to minimize an exponentially time-weighted H^∞ norm of H_{zw}, subject to steady-state tracking constraints for polynomial inputs, (ii) controller synthesis to minimize the optimally scaled H^∞ norm of H_{zw}, and (iii) controller synthesis to maximize the guaranteed dissipation of H_{zw}. It is important to note that the conditions we provide are *necessary and sufficient* when controllers are restricted to be LTI; thus our design methods provide useful *limits of performance* achievable with *any* LTI controller. The design methods also provide a parametrization of *all* LTI controllers that achieve the design goals; however, it is not easy to directly pick controllers of low or fixed order from this parametrization.

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Our approach combines interpolation theory and recent advances in numerical optimization, in particular convex optimization involving linear matrix inequalities (LMIs). We should mention that a number of other researchers have applied interpolation theory to the problem of H^∞-optimal controller design; see for example, [1, 2], [3] and [4]. Our approach differs from these in two ways. First, we show that transient-response and steady-state tracking constraints can be readily handled in the interpolation framework. Secondly, and perhaps more importantly, we show that we can compute *exact tradeoffs* between the H^∞ norm of the closed-loop map of interest and the error in the satisfaction of the input-output constraints, using convex optimization. We are not aware of any other existing methods for H^∞ controller design (two-Riccati, LMI etc.) that provide such a complete solution to these problems.

The outline of the paper is as follows. In §2, we review controller design using interpolation theory, and in §3 and §4, we discuss a number of controller design problems that can be solved using LMI-based optimization and interpolation theory.

Apart from standard notation and terminology, we also use the following. $\mathcal{D}$ denotes the open unit-disc $\{\lambda \mid \lambda \in \mathbb{C},\ |\lambda| < 1\}$. $\partial\mathcal{D}$ denotes the unit-circle $\{\lambda \mid \lambda \in \mathbb{C},\ |\lambda| = 1\}$. H^∞ denotes the set of functions that are bounded and analytic in $\mathcal{D}$. (This may differ from conventional definitions of H^∞; the reason for our convention will become apparent later.) $H^\infty_{m\times n}$ denotes the set of $m \times n$ matrices whose entries are in H^∞; such transfer matrices will be called "stable" in the sequel. For $H \in H^\infty_{m\times n}$, the H^∞ norm is defined as $\|H\|_\infty \stackrel{\Delta}{=} \sup_{|\lambda|\leq 1} \|H(\lambda)\|$, where $\|M\|$ denotes the spectral norm (maximum singular value) of a matrix M (for vectors, $\|\cdot\|$ is just the Euclidean norm). ℓ_2 denotes the Hilbert space of complex-valued square integrable sequences defined over the nonnegative integers $\mathbf{Z}_+$. For $v \in \ell_2$, $\|v\|_2$ stands for the ℓ_2-norm, defined as $\|v\|_2 = \left(\sum_{i=0}^{\infty} |v(i)|^2\right)^{1/2}$.

2 Background

Let the plant transfer matrix P be partitioned in an obvious manner as

$$P = \begin{bmatrix} P_{zw} & P_{zu} \\ P_{yw} & P_{yu} \end{bmatrix}.$$

Let $\mathcal{H}_{zw}$ denote the set of achievable stable closed-loop transfer matrices from w to z, i.e., the set of all transfer function matrices achievable over all stabilizing controllers K:

$$\mathcal{H}_{zw} = \left\{ H_{zw} \;\middle|\; H_{zw} = P_{zw} + P_{zu}K(I - P_{yu}K)^{-1}P_{yw}, \quad H_{zw} \text{ is stable} \right\}.$$

It is well-known (see for example, [5, 6]) that $\mathcal{H}_{zw}$ can alternatively be parametrized *affinely* using the Youla parameter Q as

$$\mathcal{H}_{zw} = \{T_1 - T_2QT_3 \mid Q \text{ stable}\},$$

where T_1, T_2 and T_3 are stable transfer matrices of sizes $n_z \times n_w$, $n_z \times n_u$ and $n_y \times n_w$ respectively, given explicitly in terms of the plant transfer matrix P. Moreover, every stable Q also yields a stabilizing controller K and vice versa. Thus, the problem of controller design, i.e., determining K, is equivalent to the problem of determining the Youla parameter Q.

In this paper, we will consider the special case where:

1. We have $n_y \geq n_w$ and $n_u \geq n_z$, and that $T_2(\lambda)$ and $T_3(\lambda)$ are full rank matrices for almost all λ in $\mathbb{C}$.
2. T_2 and T_3 share no zeros in $\mathcal{D}$.
3. T_2 and T_3 have no zeros on $\partial\mathcal{D}$.

Assumption (1), roughly speaking, means that we have in effect more sensors than exogenous inputs w and more actuators than controlled variables z, and that none of the actuators and sensors are redundant; thus, the problem considered is equivalent to the so-called *one-block* problem of H^∞ control. Assumptions (2) and (3) are standard technical conditions required for the application of interpolation theory; loosely speaking, these require the sensor and actuator transfer matrices to share no "nonminimum phase" zeros, and to have no zeros on the unit-circle.

Let $\alpha_1, \ldots, \alpha_p$ be the transmission zeros of T_2 in $\mathcal{D}$, with geometric multiplicities $\mu_1, \ldots, \mu_p$ respectively. The α_is are not necessarily distinct. Then, there exist vectors $u_{i,l} \in \mathbb{C}^{n_z}$, $i = 1, \ldots, p$, $l = 1, \ldots, \mu_i$ such that

$$\sum_{k=1}^{l} u_{i,k}^* \frac{1}{(l-k)!} T_2^{(l-k)}(\alpha_i) = 0, \quad l = 1, \ldots, \mu_i, \; i = 1, \ldots, p,$$

where we have used $T_2^{(l)}$ to denote the lth derivative of T_2. The set of vectors $\{u_{i,l}, \; l = 1, \ldots, \mu_i\}$ is referred to as a left-null chain of H_{zw} at α_i [7].

Let $\beta_1, \ldots, \beta_q$ be the transmission zeros of T_3 in $\mathcal{D}$, with geometric multiplicities $\nu_1, \ldots, \nu_q$ respectively. The β_is are not necessarily distinct, but they are distinct from the α_i as assumed, that is, $\alpha_i \neq \beta_l$, $i = 1, \ldots, p$, $l = 1, \ldots, q$. Then, there exist vectors $x_{i,l} \in \mathbb{C}^{n_w}$, $i = 1, \ldots, q$, $l = 1, \ldots, \nu_i$ such that

$$\sum_{k=1}^{l} \frac{1}{(l-k)!} T_3^{(l-k)}(\beta_i) x_{i,k} = 0, \quad l = 1, \ldots, \nu_i, \; i = 1, \ldots, q.$$

The set of vectors $\{x_{i,l}, \; l = 1, \ldots, \nu_i\}$ is referred to as a right-null chain of H_{zw} at β_i [7].

Since Q is stable, $H_{zw} \triangleq T_1 - T_2 Q T_3$ must satisfy

$$\sum_{k=1}^{l} u_{i,k}^* \frac{1}{(l-k)!} H_{zw}^{(l-k)}(\alpha_i) = v_{i,l}^* \triangleq \sum_{k=1}^{l} u_{i,k}^* \frac{1}{(l-k)!} T_1^{(l-k)}(\alpha_i), \quad l = 1, \ldots, \mu_i, \; i = 1, \ldots, p, \tag{1}$$

$$\sum_{k=1}^{l} \frac{1}{(l-k)!} H_{zw}^{(l-k)}(\beta_i) x_{i,k} = y_{i,l} \triangleq \sum_{k=1}^{l} \frac{1}{(l-k)!} T_1^{(l-k)}(\beta_i) x_{i,k}, \quad l = 1, \ldots, \nu_i, \; i = 1, \ldots, q. \tag{2}$$

Conversely, it can be shown that if H_{zw} satisfies (1) and (2), then there exists a stable transfer matrix Q of size $n_y \times n_u$ such that $H_{zw} = T_1 - T_2 Q T_3$. Thus (1) and (2) provide an interpolation characterization of the set $\mathcal{H}_{zw}$. Therefore, the problem of designing Q (and therefore the controller K) is equivalent to the problem of of finding H_{zw} subject to its satisfying the interpolation conditions (1) and (2). Conditions (1) and (2) are referred to as "tangential interpolation conditions" in the literature, to contrast them with matrix interpolation conditions [7].

The H^∞ norm of H_{zw} gives the "worst-case" ℓ_2-gain of H_{zw}, that is, $\|H_{zw}\|_\infty = \sup_{\|w\|_2 \leq 1} \|z\|_2$. Thus the problem of finding the smallest achievable ℓ_2-gain between w and z over all possible LTI controllers K is:

$$\text{Find } \gamma_{\rm opt} = \inf \left\{ \|H_{zw}\|_\infty \;|\; H_{zw} \text{ is stable and satisfies (1) and (2)} \right\}. \tag{3}$$

Problem (3) is the two-sided tangential Hermite-Fejér problem. Given $\gamma > 0$, a necessary and sufficient condition for $\gamma_{\rm opt} \leq \gamma$ to hold is given by the following theorem from [7, §18.5], restated here in a slightly different form for our convenience.

Theorem 1 *Given $\gamma > 0$, the quantity $\gamma_{\rm opt}$ given by (3) satisfies $\gamma_{\rm opt} \leq \gamma$ if and only if the solution N_γ to the Lyapunov equation*

$$\begin{bmatrix} I & 0 \\ 0 & \Lambda_1 \end{bmatrix} N_\gamma \begin{bmatrix} I & 0 \\ 0 & \Lambda_1 \end{bmatrix}^* - \begin{bmatrix} \Lambda_2 & 0 \\ 0 & I \end{bmatrix}^* N_\gamma \begin{bmatrix} \Lambda_2 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \gamma X^* \\ V^* \end{bmatrix} \begin{bmatrix} \gamma X & V \end{bmatrix} - \begin{bmatrix} Y^* \\ \gamma U^* \end{bmatrix} \begin{bmatrix} Y & \gamma U \end{bmatrix} \tag{4}$$

satisfies $N_\gamma \geq 0$, *where*

$$\begin{gathered}
\Lambda_1 = \mathbf{diag}\left(J^T_{\alpha_1,\mu_1}, \ldots, J^T_{\alpha_p,\mu_p}\right), \quad \Lambda_2 = \mathbf{diag}\left(J_{\beta_1,\nu_1}, \ldots, J_{\beta_q,\nu_q}\right), \\
U = [U_1 \ \cdots \ U_p], \quad V = [V_1 \ \cdots \ V_p], \quad X = [X_1 \ \cdots \ X_q], \quad Y = [Y_1 \ \cdots \ Y_q], \\
U_i = [u_{i1} \ \cdots \ u_{i\mu_i}], \quad V_i = [v_{i1} \ \cdots \ v_{i\mu_i}], \quad i = 1, \ldots, p, \\
X_i = [x_{i1} \ \cdots \ x_{i\nu_i}], \quad Y_i = [y_{i1} \ \cdots \ y_{i\nu_i}], \quad i = 1, \ldots, q.
\end{gathered} \tag{5}$$

(We have used $J_{\psi,m}$ *to denote a Jordan block of size* m *and eigenvalue* ψ*, with ones on the super-diagonal.)*

Therefore, we conclude that γ_{opt} is given by minimizing γ, subject to $N_\gamma \geq 0$, where N_γ satisfies (4). Moreover, the results in [7, §18.5] enable us to construct, for any $\gamma > \gamma_{\mathrm{opt}}$, all interpolants H_{zw} with $\|H_{zw}\|_\infty = \gamma$; this, in turn, parametrizes all stabilizing controllers that satisfy an ℓ_2-gain design constraint.

3 H^∞ controller synthesis with transient-response constraints

We now consider a variation on the standard H^∞ controller synthesis problem considered in the previous section. Consider the system shown in Figure 2. w now has the interpretation of a reference input, and d that of a disturbance.

Suppose that given a reference input w_{ref} (such as a unit-step), the transient step-response at z is required to track a given trajectory. More specifically, consider the constraint on the transient-response of the system:

$$\text{With } d = 0 \text{ and } w = w_{\mathrm{ref}}, \text{ the output } z \text{ satisfies} \quad z(0) = z_0, z(1) = z_1, \ldots, z(m-1) = z_{m-1}. \tag{6}$$

Thus, $\{z_0, \ldots, z_{m-1}\}$ is the trajectory the output is required to track over the first m time instants, when $w = w_{\mathrm{ref}}$ and there are no disturbances. Note that the output values after time $m-1$, that is, $z(m), z(m+1), \ldots$, are unconstrained. For future reference, we let $w_k = w_{\mathrm{ref}}(k)$, $k = 0, 1, \ldots, m-1$; these are just the values of the reference input over the first m time instants.

We now seek a controller that simultaneously achieves the nominal transient-response tracking, and minimizes the worst-case error energy at z due to disturbances d of bounded energy, acting at the reference input. Our new design problem is:

$$\text{Find } \gamma_{\mathrm{opt}} = \inf\left\{ \|H_{zw}\|_\infty \;\middle|\; H_{zw} \text{ is stable, satisfies (1), (2) and the transient-response condition (6)} \right\}. \tag{7}$$

Given $\gamma > 0$, we now state a necessary and sufficient condition for γ_{opt} in Problem (7) to satisfy $\gamma_{\mathrm{opt}} \leq \gamma$.

Proposition 1 *Given* $\gamma > 0$*, the quantity* γ_{opt} *given by (7) satisfies* $\gamma_{\mathrm{opt}} \leq \gamma$ *if and only if the solution* N_γ *to the Lyapunov equation*

$$\begin{bmatrix} I & 0 \\ 0 & \Lambda_1 \end{bmatrix} N_\gamma \begin{bmatrix} I & 0 \\ 0 & \Lambda_1 \end{bmatrix}^* - \begin{bmatrix} \tilde{\Lambda}_2 & 0 \\ 0 & I \end{bmatrix}^* N_\gamma \begin{bmatrix} \tilde{\Lambda}_2 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \gamma \tilde{X}^* \\ V^* \end{bmatrix} \begin{bmatrix} \gamma \tilde{X} & V \end{bmatrix} - \begin{bmatrix} \tilde{Y}^* \\ \gamma U^* \end{bmatrix} \begin{bmatrix} \tilde{Y} & \gamma U \end{bmatrix} \tag{8}$$

satisfies $N_\gamma \geq 0$, *where*

$$\begin{gathered}
\tilde{\Lambda}_2 = \mathbf{diag}\left(J_{\beta_1,\nu_1}, \ldots, J_{\beta_q,\nu_q}, J_{\beta_{q+1},\nu_{q+1}}\right), \quad \beta_{q+1} = 0, \quad \nu_{q+1} = m, \quad \tilde{X} = [X \ \ X_{q+1}], \quad \tilde{Y} = [Y \ \ Y_{q+1}], \\
X_{q+1} = [w_0 \ \cdots \ w_{m-1}], \quad Y_{q+1} = [z_0 \ \cdots \ z_{m-1}],
\end{gathered} \tag{9}$$

and Λ_1, X, Y, U, V *are defined in (5).*

Proof: We first show that the transient-response condition (6) is equivalent to a number of additional interpolation conditions on H_{zw}.

Let $H_{zw}(\lambda) = \sum_{i=0}^{\infty} h_i \lambda^i$. Note that this is different from the conventional definition of the z-transform which is a power series in z^{-1}, but consistent with our definition of H^∞. Since $h_k = \frac{1}{k!} H_{zw}^{(k)}(\lambda)\Big|_{\lambda=0}$, the transient-response constraints imply that

$$\begin{bmatrix} z_0 \\ z_1 \\ \vdots \\ z_{m-1} \end{bmatrix} = \begin{bmatrix} H_{zw}(0) & 0 & \cdots & 0 \\ \frac{1}{1!} H_{zw}^{(1)}(0) & H_{zw}(0) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{(m-1)!} H_{zw}^{(m-1)}(0) & \frac{1}{(m-2)!} H_{zw}^{(m-2)}(0) & \cdots & H_{zw}(0) \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \\ \vdots \\ w_{m-1} \end{bmatrix}. \tag{10}$$

Thus, the transient-response constraints just lead to additional tangential interpolation conditions on H_{zw} at the origin:

$$\sum_{k=1}^{j} \frac{1}{(j-k)!} H_{zw}^{(j-k)}(0) w_{k-1} = z_{j-1}, \qquad j = 1, \ldots, m. \tag{11}$$

Introducing the notation $\beta_{q+1} = 0$ and $\nu_{q+1} = m$, adding these interpolation conditions to (1) and (2), and applying Theorem 1, concludes the proof.

□

We note that we may have multiple transient-response constraints for multiple reference inputs; these lead to multiple interpolation constraints at the origin.

3.1 Approximate transient-response constraints

Next, consider γ satisfying $\gamma < \gamma_{\mathrm{opt}}$, where γ_{opt} is given by (7). Then, there exists no LTI controller that achieves exact transient-response tracking subject to $\|H_{zw}\|_\infty \leq \gamma$ (we will refer to γ as the "disturbance attenuation level"). In this case, we may be willing to relax the exact transient-response constraint, by accepting a nonzero tracking error measured in some appropriate manner. We now show that the study of the tradeoff between the transient-response tracking error and the disturbance attenuation level can be efficiently performed using numerical convex optimization. We will thus be able to answer questions such as, "What is the smallest transient-response tracking error, over all stabilizing controllers K, for a given disturbance attenuation level γ?" Our answers will be global optima, and thus we will be studying *exact* tradeoffs, with LTI controllers K, between tracking and disturbance rejection.

With $d = 0$ and $w = w_{\mathrm{ref}}$, we allow the corresponding output z to lie in the set

$$\left\{ \tilde{z} \,\middle|\, \tilde{z}(0) = z_0 + e_0, \tilde{z}(1) = z_1 + e_1, \ldots, \tilde{z}(m-1) = z_{m-1} + e_{m-1},\ e_k^T e_k \leq \epsilon^2,\ k = 0, \ldots, m-1 \right\}. \tag{12}$$

The output values after time $m-1$, that is, $z(m), z(m+1), \ldots$, are unconstrained as before. The left-half of Figure 3 shows such a transient-response tracking constraint over the first five time instants, for some reference input w_{ref} (this input is not shown in the figure). The solid lines show the desired nominal response at z, and dotted lines show the transient-response tracking error allowed.

The quantity ϵ in (12) is a measure of the transient-response tracking error. In this case, the interpolation conditions (11) become

$$\sum_{k=1}^{j} \frac{1}{(j-k)!} H_{zw}^{(j-k)}(0) w_{k-1} = z_{j-1} + e_{j-1}, \quad j = 1, \ldots, m, \text{ with } e_{j-1}^T e_{j-1} \leq \epsilon^2. \tag{13}$$

From Proposition 1 and straightforward algebraic manipulations, we conclude the following:

Corollary 1 *Given a transient-response tracking error ϵ and a disturbance attenuation level γ, there exists an LTI controller K achieving transient-response tracking with error not exceeding ϵ and a disturbance attenuation level not exceeding γ, if and only if there exists a matrix E satisfying*

$$E = [0 \ \cdots \ 0 \ \ E_{q+1}], \ \ \textit{with} \ E_{q+1} = [e_0 \cdots e_{m-1}], \quad e_j^T e_j \leq \epsilon^2, \ j = 0, \ldots, m-1, \tag{14}$$

such that

$$N_\gamma \geq 0, \ \textit{and} \ \begin{bmatrix} I & 0 \\ 0 & \Lambda_1 \end{bmatrix} N_\gamma \begin{bmatrix} I & 0 \\ 0 & \Lambda_1 \end{bmatrix}^* - \begin{bmatrix} \tilde{\Lambda}_2 & 0 \\ 0 & I \end{bmatrix}^* N_\gamma \begin{bmatrix} \tilde{\Lambda}_2 & 0 \\ 0 & I \end{bmatrix} \tag{15}$$
$$= \begin{bmatrix} \gamma \tilde{X}^* \\ V^* \end{bmatrix} \begin{bmatrix} \gamma \tilde{X} & V \end{bmatrix} - \begin{bmatrix} (\tilde{Y}+E)^* \\ \gamma U^* \end{bmatrix} \begin{bmatrix} \tilde{Y}+E & \gamma U \end{bmatrix},$$

where the various quantities in the above equation are defined in (5) and (9).

At first glance, given γ and ϵ, checking whether there exists E such that conditions (14) and (15) hold may appear a formidable numerical task. However, we have the following theorem that simplifies matters considerably.

Theorem 2 *For fixed γ, the set of variables (ϵ^2, E) such that conditions (14) and (15) hold is a convex set described by linear matrix equations and linear matrix inequalities.*[1]

Proof: It can be shown from simple monotonicity properties of solutions to Lyapunov equations that for some E satisfying (14), the matrix N_γ given by the Lyapunov equation in (15) satisfies $N_\gamma \geq 0$, if and only if for some $W \geq 0$ and $R \leq 0$, the following hold:

$$\begin{bmatrix} I & 0 \\ 0 & \Lambda_1 \end{bmatrix} W \begin{bmatrix} I & 0 \\ 0 & \Lambda_1 \end{bmatrix}^* - \begin{bmatrix} \tilde{\Lambda}_2 & 0 \\ 0 & I \end{bmatrix}^* W \begin{bmatrix} \tilde{\Lambda}_2 & 0 \\ 0 & I \end{bmatrix}$$
$$- \begin{bmatrix} \gamma \tilde{X}^* \\ V^* \end{bmatrix} \begin{bmatrix} \gamma \tilde{X} & V \end{bmatrix} + \begin{bmatrix} (\tilde{Y}+E)^* \\ \gamma U^* \end{bmatrix} \begin{bmatrix} \tilde{Y}+E & \gamma U \end{bmatrix} = \begin{bmatrix} R & 0 \\ 0 & 0 \end{bmatrix}.$$

These conditions, using standard manipulations, can be shown to be equivalent to the matrix inequalities and equations

$$W = \begin{bmatrix} W_{11} & W_{12} \\ W_{12}^* & W_{22} \end{bmatrix} \geq 0, \quad \begin{bmatrix} W_{11} - \tilde{\Lambda}_2^* W_{11} \tilde{\Lambda}_2 - \gamma^2 \tilde{X}^* \tilde{X} & (\tilde{Y}+E)^* \\ (\tilde{Y}+E) & -I \end{bmatrix} \leq 0,$$
$$W_{12}\Lambda_1^* - \tilde{\Lambda}_2^* W_{12} - \gamma \left(\tilde{X}^* V - (\tilde{Y}+E)^* U \right) = 0, \quad \Lambda_1 W_{22} \Lambda_1^* - W_{22} + \gamma^2 U^* U - V^* V = 0. \tag{16}$$

Finally, the constraint that E satisfies (14) is equivalent to

$$E = [0 \ \cdots \ 0 \ \ E_{q+1}], \ \begin{bmatrix} F & E^T \\ E & I \end{bmatrix} \geq 0, \quad F_{ii} \leq \epsilon^2, \quad i = 1, \ldots, \sum_{i=1}^{q+1} \nu_i, \tag{17}$$

where F_{ii} denotes the ith diagonal element of F.

Therefore, given $\gamma > 0$, there exist ϵ and E such that conditions (14) and (15) hold, if and only if there exist ϵ, E, F and W, satisfying (16) and (17); these latter conditions are linear matrix equations and inequalities in the variables E, F, W, and ϵ^2, and the statement of the theorem follows immediately.

□

Theorem 2 has important implications. For example, given γ and ϵ, checking whether there exists E such that conditions (14) and (15) hold is a *convex* feasibility problem, in particular, an LMI feasibility problem. Moreover, given a disturbance level γ, the smallest value of ϵ for which there exists E such that conditions (14) and (15) hold can be reformulated as a convex LMI optimization problem. Consequently, these problems can be numerically solved very reliably and efficiently, using algorithms that rapidly compute the global optima, with non-heuristic stopping criteria and guaranteed convergence. We refer the reader to [8, 9] for details about LMI optimization problems and their numerical solution.

[1] A linear matrix inequality or LMI is a matrix inequality of the form $F(x) = F_0 + \sum_{i=1}^{l} x_i F_i \geq 0$, where $x_1, x_2, \ldots, x_l$ are the variables, $F_i = F_i^T \in \mathbf{R}^{n \times n}$ are given, and $F(x) \geq 0$ means that $F(x)$ is positive semi-definite.

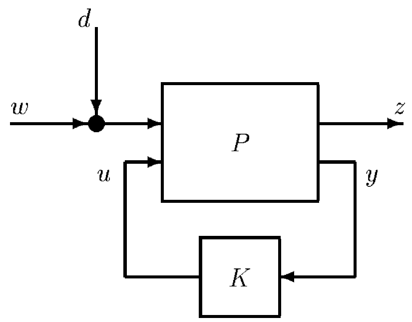

Figure 2: Controller design with input-output constraints

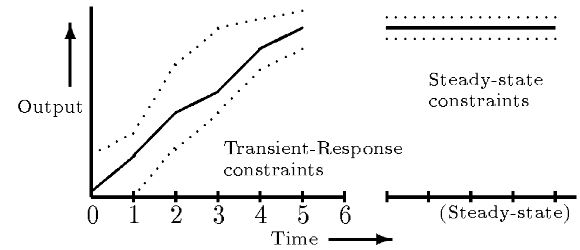

Figure 3: Example of transient and steady-state constraints on the output; a single-output system is shown here.

4 Some extensions

4.1 H^∞ controller synthesis with steady-state constraints

In the previous section, we addressed the issue of H^∞-optimal control subject to transient-response tracking. We now consider steady-state tracking for polynomial-like inputs, i.e., inputs that are linear combination of polynomial sequences $w_m : \mathbf{Z}_+ \to \mathbf{R}$ given by

$$w_m(k) = \begin{cases} 1 & \text{if } m = 0 \\ (k+1)\cdots(k+m) & \text{if } m > 0. \end{cases} \tag{18}$$

For example, with x_0, x_1 being constant vectors in $\mathbf{R}^{n_w}$, $w_0 x_0$ is a (multi-input) step, and $w_1 x_1$ a ramp.

Consider a steady-state tracking constraint of order m, of the form:

For the input $w = w_0 x_0 + w_1 x_1 + w_2 x_2 + \cdots + w_{m-1} x_{m-1}$,
the output is z satisfies $\lim_{k\to\infty} \|z(k) - (w_0(k)y_0 + w_1(k)y_1 + w_2(k)y_2 + \cdots + w_{m-1}(k)y_{m-1})\| = 0$,
(19)

where $x_i \in \mathbf{R}^{n_w}$ and $y_i \in \mathbf{R}^{n_z}$, $i = 0, \ldots, m-1$ are specified constants. It is an easy exercise to verify that these constraints lead to the following interpolation conditions on the closed-loop map H_{zw} at $\lambda = 1$.

$$\begin{bmatrix} y_{m-1} \\ y_{m-2} \\ \vdots \\ y_0 \end{bmatrix} = \begin{bmatrix} H_{zw}(1) & 0 & \cdots & 0 \\ \frac{(-1)}{1!} H_{zw}^{(1)}(1) & H_{zw}(1) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \frac{(-1)^{m-1}}{(m-1)!} H_{zw}^{(m-1)}(1) & \frac{(-1)^{m-2}}{(m-2)!} H_{zw}^{(m-2)}(1) & \cdots & H_{zw}(1) \end{bmatrix} \begin{bmatrix} x_{m-1} \\ x_{m-2} \\ \vdots \\ x_0 \end{bmatrix}. \tag{20}$$

Once again, steady-state response constraints lead to a number of additional tangential interpolation condition on H_{zw}, as with transient-response constraints. However, there is an important difference: The additional interpolation conditions (20) are *on* the unit-circle $\partial\mathcal{D}$. Therefore, the problem of finding the smallest achievable ℓ_2-gain between w and z, that is, finding

$$\gamma_{\text{opt}} = \inf\{\|H_{zw}\|_\infty \mid H_{zw} \text{ is stable, satisfies (1), (2) and (20)}\}$$

is *not* a standard two-sided tangential interpolation problem.

Therefore, we consider an alternate problem: We minimize, for some $\rho > 1$, the ρ-shifted H^∞ norm of H_{zw}, defined as $\|H_{zw}\|_{\infty,\rho} \stackrel{\Delta}{=} \sup_{|\lambda|\le\rho} \|H_{zw}(\lambda)\|$. This is equivalent to introducing an additional "stability-degree" constraint; see [10]. The choice of ρ, as well as the properties of the solution of the design problem as $\rho \to 1$ are of independent interest; we will not investigate these questions here.

The modified design problem is then:

$$\text{Find } \gamma_{\text{opt}} = \inf\left\{ \|H_{zw}\|_{\infty,\rho} \mid H_{zw} \text{ is stable, satisfies (1), (2) and (20)} \right\}. \tag{21}$$

Note that $\|H_{zw}\|_{\infty,\rho} \geq \|H_{zw}\|_{\infty}$ for all $\rho \geq 1$, so we are in effect minimizing an upper bound on the ℓ_2-gain, subject to tracking constraints, for the original system.

Proposition 2 *Problem (21) is a standard two-sided interpolation problem such as Problem (3).*

Proof: Consider the exponentially time-weighted system $\tilde{H}_{zw}(\lambda) = H_{zw}(\rho\lambda)$ for some fixed $\rho > 1$. Then, $\|H_{zw}\|_{\infty,\rho} = \|\tilde{H}_{zw}\|_{\infty}$. Moreover, all the interpolation conditions are in $\partial\mathcal{D} \bigcup \mathcal{D}$ for H_{zw}, and therefore they are all in $\mathcal{D}$ for $\tilde{H}_{zw}$. This concludes the proof.

□

Combining Proposition 2 and Theorem 1, we can immediately solve the problem of controller synthesis to minimize the ρ-shifted H^{∞} norm of H_{zw}, subject to steady-state tracking of polynomial reference inputs. We can also incorporate, in a straightforward manner, steady-state tracking constraints for discrete-time *sinusoids* using the techniques outlined in this section; these constraints also lead to interpolation conditions on the unit-circle. We may consider simultaneous transient-response and steady-state tracking as well.

We may also require the system to track polynomial inputs approximately, that is, it may be desired that for the polynomial input

$$\begin{array}{l} w = w_0 x_0 + w_1 x_1 + w_2 x_2 + \cdots + w_{m-1} x_{m-1}, \\ \text{the response of the system satisfy } \lim_{k\to\infty} \|z(k) - \tilde{z}(k)\| = 0 \text{ for some } \tilde{z} \text{ in the set} \\ \left\{ w_0(y_0 + e_0) + w_1(y_1 + e_1) + \cdots + w_{m-1}(y_{m-1} + e_{m-1}) \mid e_i^T e_i \leq \epsilon^2,\ i = 0, \ldots, m-1 \right\}. \end{array} \tag{22}$$

The right-half of Figure 3 shows a simple step-tracking constraint. The solid lines show the steady-state response at z with exact tracking, and the dotted lines show the steady-state tracking error allowed.

As in §3.1, we may apply interpolation theory and LMI-based convex optimization and study tradeoffs between the ρ-shifted H^{∞} norm for the closed-loop system and steady-state tracking error for polynomial inputs.

4.2 Minimizing the scaled ℓ_2-gain

We mention another interpolation problem that can be solved by combining LMI optimization and interpolation theory. We consider the special case when H_{zw} is square, that is, $n_w = n_z$, and when T_2 has no zeros in $\mathcal{D}$, so that (1) is a vacuous constraint. For this case, we consider the problem of minimizing the *diagonally scaled* ℓ_2-gain between w and z over all possible LTI controllers K, subject to transient-response tracking constraints of the type (10), that is:

$$\text{Find } \gamma_{\text{opt}} = \inf\left\{ \|DH_{zw}D^{-1}\|_{\infty} \;\middle|\; \begin{array}{l} H_{zw} \text{ stable, satisfies (2), (11),} \\ D \in \mathbf{R}^{n\times n}, \text{ diagonal, nonsingular} \end{array} \right\}. \tag{23}$$

We remark that the discussion that follows applies to the more general case when D is block-diagonal, with prespecified block sizes. This problem arises in H^{∞} control of systems with structured perturbations [11, 12]. A similar problem (without any input-output constraints) is considered by Safonov [13], who uses matrix interpolation theory, as opposed to the tangential interpolation theory that is presented here.

Proposition 3 *Given $\gamma > 0$, the quantity γ_{opt} given by (23) satisfies $\gamma_{\text{opt}} \leq \gamma$ if and only if there exists a diagonal matrix $S > 0$ such that the solution N_γ to the Lyapunov equation*

$$N_\gamma - \tilde{\Lambda}_2^* N_\gamma \tilde{\Lambda}_2 = \left(\gamma^2 \tilde{X}^* S \tilde{X} - \tilde{Y}^* S \tilde{Y}\right) \tag{24}$$

satisfies $N_\gamma \geq 0$, where $\tilde{\Lambda}_2$, $\tilde{X}$ and $\tilde{Y}$ are as defined in equation (9).

Proof: We first note that for any scaling D,

$$\sum_{k=1}^{j} \frac{1}{(j-k)!} H_{zw}^{(j-k)}(\beta_i) x_{i,k} = y_{i,j} \text{ is equivalent to } \sum_{k=1}^{j} \frac{1}{(j-k)!} (D H_{zw} D^{-1})^{(j-k)}(\beta_i) D x_{i,k} = D x_{i,j},$$

and therefore, from Proposition 1, there exists a scaling D such that $\|D H_{zw} D^{-1}\|_\infty < \gamma$ with H_{zw} satisfying (2) and (11) if and only if the following hold:

$$N_\gamma \geq 0, \quad N_\gamma - \tilde{\Lambda}_2^* N_\gamma \tilde{\Lambda}_2 = \left(\gamma^2 \tilde{X}^* S \tilde{X} - \tilde{Y}^* S \tilde{Y} \right),$$

where $S = D^2$ is diagonal and positive definite, and $\tilde{\Lambda}_2$, $\tilde{X}$ and $\tilde{Y}$ are as defined in equation (9).

□

Thus, the solution to problem (23) may be found by solving a quasi-convex LMI optimization problem—the so-called generalized eigenvalue problem or GEVP (see [9]):

Minimize γ^2, over the variables diagonal $S > 0$, N_γ and γ^2, subject to (24).

Using a similar approach, we may readily solve variations of Problem (23), for example, with steady-state tracking constraints and the ρ-shifted H^∞ norm objective.

4.3 Positive real interpolation

We may extend all the forementioned results to a related problem, that of designing K to maximize the guaranteed dissipation η of the closed-loop map H_{zw}, which is defined as

$$\eta(H_{zw}) \triangleq \inf_{\lambda \in \mathcal{D}} \chi_{\min} \left(\frac{H_{zw}(\lambda) + H_{zw}(\lambda)^*}{2} \right),$$

where $\chi_{\min}(M)$ stands for the minimum eigenvalue of the Hermitian matrix M. If w and z are power conjugate quantities (i.e., their inner-product has the interpretation of power), $\eta(H_{zw})$ can be thought of as the "dissipation factor". Thus, the larger η is, the larger the dissipation, and it is often of interest in circuit theory [14, Ch. 9] to design systems so as to maximize the dissipation factor.

Consider then, the basic design problem (without any input-output constraints, for simplicity):

$$\text{Find } \eta_{\text{opt}} = \sup \{ \eta(H_{zw}) \mid H_{zw} \text{ is stable and satisfies (1) and (2)} \}. \tag{25}$$

The solution of this problem proceeds via the following proposition.

Proposition 4 *Given δ, the quantity η_{opt} given by (25) satisfies $\eta_{\text{opt}} \geq \delta$ if and only if the solution N to the Lyapunov equation*

$$\begin{aligned} &\begin{bmatrix} I & 0 \\ 0 & \Lambda_1 \end{bmatrix} N \begin{bmatrix} I & 0 \\ 0 & \Lambda_1 \end{bmatrix}^* - \begin{bmatrix} \Lambda_2 & 0 \\ 0 & I \end{bmatrix}^* N \begin{bmatrix} \Lambda_2 & 0 \\ 0 & I \end{bmatrix} \\ &= 2 \begin{bmatrix} -2\delta X^* X + X^* Y + Y^* X & Y^* U - X^* V \\ U^* Y - V^* X & 2\delta U^* U - U^* V - V^* U \end{bmatrix}, \end{aligned} \tag{26}$$

satisfies $N \geq 0$, where

$$\hat{U} = (1-\delta) U + V, \quad \hat{V} = (1+\delta) U - V, \quad \hat{X} = (1-\delta) X + Y, \quad \hat{Y} = (1+\delta) X - Y, \tag{27}$$

and U, V, X, Y, Λ_1 and Λ_2 are defined in (5).

Proof: Our proof hinges on reformulating Problem (25) as an ℓ_2-gain minimization problem for some other system, under different interpolation constraints. Let $G_{zw} = (I + (H_{zw} - \delta I))^{-1}(I - (H_{zw} - \delta I))$. Then, it is easily verified that H_{zw} satisfies (1) and (2) if and only if G_{zw} satisfies the interpolation conditions

$$\sum_{k=1}^{l} \left((1-\delta)u_{i,k}^* + v_{i,k}^*\right) \frac{1}{(l-k)!} G_{zw}^{(l-k)}(\alpha_i) = (1+\delta)u_{i,l}^* - v_{i,l}^*,\ l = 1,\ldots,\delta_i,\ i = 1,\ldots,p, \tag{28}$$

$$\sum_{k=1}^{l} \frac{1}{(l-k)!} G_{zw}^{(l-k)}(\beta_i)\left((1-\delta)x_{i,k} + y_{i,k}\right) = (1+\delta)x_{i,l} - y_{i,l},\ l = 1,\ldots,\nu_i,\ i = 1,\ldots,q. \tag{29}$$

Moreover, H_{zw} satisfies $\eta(H_{zw}) \geq \delta$ if and only if $\|G_{zw}\|_\infty \leq 1$.

Therefore, we conclude using Theorem 1 that there exists a controller such that $\eta(H_{zw}) \geq \delta$ if and only if the solution N to the Lyapunov equation

$$\begin{bmatrix} I & 0 \\ 0 & \Lambda_1 \end{bmatrix} N \begin{bmatrix} I & 0 \\ 0 & \Lambda_1 \end{bmatrix}^* - \begin{bmatrix} \Lambda_2 & 0 \\ 0 & I \end{bmatrix}^* N \begin{bmatrix} \Lambda_2 & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \hat{X}^* \\ \hat{V}^* \end{bmatrix} \begin{bmatrix} \hat{X} & \hat{V} \end{bmatrix} - \begin{bmatrix} \hat{Y}^* \\ \hat{U}^* \end{bmatrix} \begin{bmatrix} \hat{Y} & \hat{U} \end{bmatrix} \tag{30}$$

satisfies $N \geq 0$, where $\hat{U}$, $\hat{V}$, $\hat{X}$ and $\hat{Y}$ are given by (27), and U, V, X, Y, Λ_1 and Λ_2 are defined in (5).

After algebraic manipulations, the Lyapunov equation (30) simplifies to (26), completing the proof. □

We can extend the results of §3–4.2 in a straightforward manner to the problem designing K to maximize the guaranteed dissipation, subject to transient and steady-state input-output constraints.

5 A numerical example

We consider an example with

$$P(\lambda) = \begin{bmatrix} P_{zw}(\lambda) & P_{zu}(\lambda) \\ P_{yw}(\lambda) & P_{yu}(\lambda) \end{bmatrix} = \left[\begin{array}{cc|c} 1 & \frac{1}{\lambda-1.5} & \\ \frac{1}{\lambda-1.5} & \frac{\lambda-1.5}{\lambda-2.5} & -I \\ \hline 1 & \frac{1}{\lambda-1.5} & \\ 0 & \frac{(\lambda-0.5)^2}{(\lambda+2)^2} & 0 \end{array}\right] = \begin{bmatrix} T_1(\lambda) & -T_2(\lambda) \\ T_3(\lambda) & 0 \end{bmatrix}. \tag{31}$$

The interpolation conditions on H_{zw} are:

$$H_{zw}(0.5)\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.5 \end{bmatrix}, \quad H_{zw}^{(1)}(0.5)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + H_{zw}(0.5)\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1.75 \end{bmatrix}. \tag{32}$$

Since $T_2(\lambda) = I$, there are no left interpolation conditions on H_{zw}. Thus, we have, in the notation of (5),

$$\Lambda_2 = \begin{bmatrix} 0.5 & 1 \\ 0 & 0.5 \end{bmatrix}, \quad X = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 \\ -0.5 & -1.75 \end{bmatrix}.$$

Using Theorem 1, we first determine that the smallest value of $\|H_{zw}\|_\infty$ subject to H_{zw} satisfying the interpolation conditions (32) is 1.0999. For future reference, we denote this value by $\gamma_{\text{opt,free}}$ (the optimal disturbance attenuation level, free of any tracking constraints).

We next impose a transient-response constraint, requiring that for the reference input whose values over the first five time instants are

$$\left(\begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 1.0 \end{bmatrix}, \begin{bmatrix} 0.0 \\ 0.5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ -0.5 \end{bmatrix}, \begin{bmatrix} 1.0 \\ 0.0 \end{bmatrix}\right), \tag{33}$$

the output for the first five time instants is

$$\left(\begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix}, \begin{bmatrix} -0.5 \\ 1.0 \end{bmatrix}, \begin{bmatrix} 0.0 \\ 0.5 \end{bmatrix}, \begin{bmatrix} 0.5 \\ 1.0 \end{bmatrix}, \begin{bmatrix} -1.0 \\ 1.0 \end{bmatrix} \right). \tag{34}$$

(It is unconstrained thereafter.) Using Proposition 1, we solve for the smallest value of $\|H_{zw}\|_\infty$ subject to the additional interpolation conditions imposed by the exact transient-response tracking constraint. The answer is 10.6917, and we denote this by $\gamma_{\rm opt,0}$ (the optimal disturbance attenuation level, subject to zero tracking error).

We then relax these constraints by allowing a transient-response tracking error of ϵ (see (12)). The tradeoff between the transient-response tracking error ϵ and the corresponding smallest achievable $\|H_{zw}\|_\infty$, denoted $\gamma_{\rm opt,\epsilon}$, is plotted in Figure 4. As expected, when $\gamma_{\rm opt,\epsilon} \geq 10.6917$, which is the smallest achievable $\|H_{zw}\|_\infty$ subject to exact transient-response tracking, $\epsilon = 0$. As $\gamma_{\rm opt,\epsilon}$ becomes smaller than $\gamma_{\rm opt,0}$, the tracking error ϵ grows larger. When $\gamma_{\rm opt,\epsilon}$ becomes smaller than the smallest achievable $\|H_{zw}\|_\infty$ subject to no transient-response constraints, i.e., $\gamma_{\rm opt,free}$, the problem becomes infeasible (i.e., the tracking error ϵ jumps to ∞). Thus, the tradeoff curve is *discontinuous* at $\gamma_{\rm opt,\epsilon} = \gamma_{\rm opt,free}$.

Next, we impose the following steady-state tracking constraint, in the place of the transient-response constraints:

$$\text{For the input} \quad w_0 \begin{bmatrix} 1.5 \\ -1.0 \end{bmatrix} + w_1 \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix}, \text{ the output } z \text{ satisfies}$$
$$\lim_{k\to\infty} \left\| z(k) - w_0(k) \begin{bmatrix} 0.5 \\ -1.0 \end{bmatrix} - w_1(k) \begin{bmatrix} 1.0 \\ 1.0 \end{bmatrix} \right\| = 0. \tag{35}$$

(Recall that w_0 is the unit-step and w_1 the unit-ramp.) With these additional interpolation constraints, the smallest achievable exponentially time-weighted H^∞ norm (with weighting 1.1) for H_{zw} is 4.1570.

Relaxing these tracking constraints allows for lower values for $\|H_{zw}\|_{\infty,1.1}$. The tradeoff between the steady-state tracking error ϵ, defined in (22), and the smallest achievable value of $\|H_{zw}\|_{\infty,1.1}$ is plotted in Figure 5. We can make similar statements here, as with tradeoff curve in Figure 4.

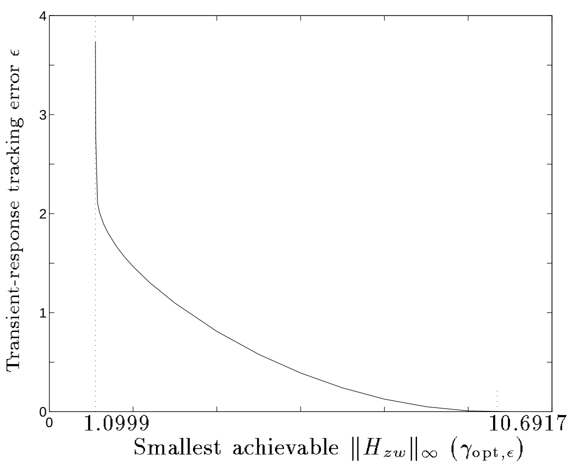

Figure 4: Tradeoff between the smallest achievable $\|H_{zw}\|_\infty$ and the transient-response tracking error ϵ.

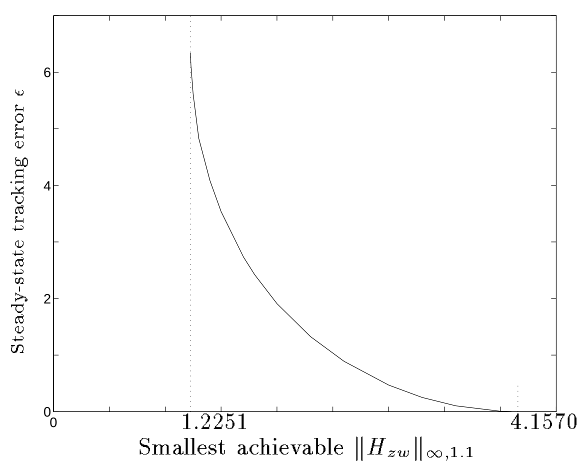

Figure 5: Tradeoff between the smallest achievable $\|H_{zw}\|_{\infty,1.1}$ and the steady-state tracking error ϵ.

6 Conclusions

By combining interpolation theory with recent advances in LMI-based convex optimization, we have shown how we may solve a number of important controller design problems. We have demonstrated our results with a simple example. The results presented here also serve to extend the results on system identification, presented in [15] for the SISO case, to the case of multi-input multi-output systems.

Based on the results herein, it is easy to study tradeoffs between the ℓ_2 gain (or the guaranteed dissipation) of the closed-loop system and how accurately the closed-loop system satisfies various input-output constraints. Computer-aided controller design tools that exploit these results can be devised in a straightforward manner.

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