Linear Matrix Inequalities in Robustness Analysis with Multipliers[*]

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Abstract

We show that a number of standard robustness tests can be reinterpreted as special cases of the application of the passivity theorem with the appropriate choice of multipliers. We show how these tests can be performed using convex optimization over linear matrix inequalities.

Keywords: Robustness analysis, passivity theorem, multipliers, convex optimization, linear matrix inequalities.

1 Notation

The notation and terminology are standard; for details and precise technical conditions, we refer the reader to the book by Desoer and Vidyasagar [7].

$\mathbf{L}_m^2$ is the Hilbert space of square-integrable signals defined over $[0, \infty)$, with m components. A causal m-input m-output operator L is said to be $\mathbf{L}_m^2$ *stable* or just *stable* if there exist $\gamma \geq 0$ and β such that

$$||Lu||_2 \leq \gamma ||u||_2 + \beta \quad \forall u \in \mathbf{L}_m^2. \tag{1}$$

The smallest γ such that (1) holds for some β is called the $\mathbf{L}_m^2$ *gain* of L.

L is said to be *strictly passive* if it satisfies, for some real β and $\epsilon > 0$,

$$\int_0^T u(t)^T (Lu)(t)\, dt \geq \epsilon \int_0^T u(t)^T u(t)\, dt + \beta \text{ for all } T \geq 0 \text{ and all } u \in \mathbf{L}_m^2. \tag{2}$$

It is *passive* if it satisfies (2) with $\epsilon \geq 0$.

The $\mathbf{H}_\infty$ norm of the transfer matrix H of a causal, linear time-invariant system $\mathcal{H}$ is denoted $||H||_\infty$, and defined as $||H||_\infty = \sup_{\mathrm{Re}\, s > 0} \sigma_{\max}(H(s))$, where $\sigma_{\max}(M)$ is the maximum singular value of the matrix M. Of course, $\mathcal{H}$ is stable if and only if $||H||_\infty$ is finite, in which case $||H||_\infty$ equals the $\mathbf{L}_m^2$ gain of $\mathcal{H}$.

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The matrix inequalities $A > B$ and $A \geq B$ mean A and B are square, Hermitian, and that $A - B$ is positive definite and positive semi-definite, respectively.

2 Introduction

A large class of robust control problems are posed in the setting shown in Figure 1. $\mathcal{H}$ is a causal, finite-dimensional, linear time-invariant (LTI), stable system with m inputs and m outputs. We let $H(s)$ denote the transfer matrix of $\mathcal{H}$. The perturbation Δ, which in general is a nonlinear operator, appears in the feedback loop; it might represent nonlinearities and dynamics that are either unknown, unmodeled or neglected. We assume that Δ is causal and stable. Note that there are no external inputs or outputs. We will refer to the system shown in Figure 1 as $\mathcal{L}$.

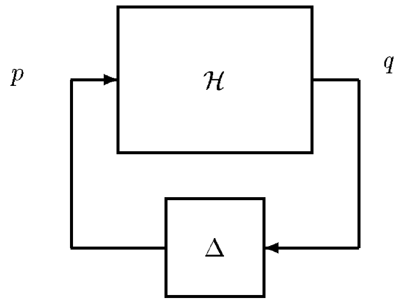

Figure 1: System $\mathcal{L}$: A standard framework for robustness analysis

We say that system $\mathcal{L}$ is $\mathbf{L}_m^2$-stable (or just "stable") if the signals p and q belong to $\mathbf{L}_m^2$.

Often information about the "size" of Δ is available; most commonly, the $\mathbf{L}_m^2$ gain of Δ is known or assumed to be less than or equal to some positive γ. In this case, the small gain theorem [7] states that the system $\mathcal{L}$ is stable for all Δ (this is also known as *robust* stability) if the $\mathbf{H}_\infty$ norm of H is less than $1/\gamma$. Thus, the *stability margin* of the system, defined as the largest size of the perturbations against which the system is robustly stable, is at least $1/\|H\|_\infty$.

Often, additional information is known or assumed about Δ: Δ is diagonal or block-diagonal (i.e., "structured"); Δ is a convolution operator (i.e., "LTI perturbation"); Δ is a constant matrix (i.e., "parametric"), etc. In these cases, a host of sufficient conditions for robust stability are available, each derived using apparently very different techniques. The first objective of this paper is to show that several of these sufficient conditions for robust stability can be derived in a single framework, that of the *passivity theorem with multipliers.* We should point out at the outset that several other researchers have noted parts of what we will show; in particular, Safonov and Chiang, in their work on real/complex K_m synthesis using multipliers [11], note that their analysis method in the case of LTI perturbations Δ is equivalent to the frequency-dependent D scaling of μ analysis [6]. They also remark that their method is more powerful in the case of parametric uncertainties Δ. We show that their analysis method is in fact *equivalent* to the off-axis circle stability criterion for systems with real parameters, due to Fan, Tits and Doyle [8]. How and Hall [9] have rederived this stability criterion using dissipation theory (which is equivalent to passivity analysis with multipliers). We present these results nevertheless, since our derivation shows how several robust stability tests can be derived in a unified manner in one setting; also, our derivation is essential for the second objective of this paper, which is to show how these tests can be performed using convex optimization over linear matrix inequalities using state-space techniques. This second contribution, we believe, is entirely new.

The organization of the paper is as follows. In Section 3, we consider various robustness tests for system $\mathcal{L}$, and show how they can be rederived using the passivity theorem with multipliers. In Section 4, we show how these tests can be reduced to convex feasibility problems, in particular, feasibility of linear matrix inequalities. In Section 5, we present a simple numerical example.

3 Robustness tests using the passivity theorem with multipliers

We will first transform system $\mathcal{L}$ to the framework of the passivity theorem. To this end, we will apply a simple linear fractional transformation to the system by defining new variables

$$\tilde{p} = \gamma q - p, \qquad \tilde{q} = \gamma q + p.$$

(Recall that γ is the upper bound on the $\mathbf{L}_m^2$ gain of Δ.) This transformation is well-known in the literature; see for example [1] and [7, Section VI.10]. It is referred to as the "bilinear sector transformation" in [11].

Suppose $(\gamma I + \Delta)$ is invertible[1]. Then, after routine algebraic manipulations, system $\mathcal{L}$ can be rewritten as system $\tilde{\mathcal{L}}$ shown in Figure 2, where $\mathcal{G}$ is an LTI system with transfer matrix $G(s)$ given by $G(s) = (I - \gamma H(s))^{-1} (I + \gamma H(s))$, and $\tilde{\Delta} = (\gamma I - \Delta) \circ (\gamma I + \Delta)^{-1}$. (The symbol "$\circ$" denotes composition.) It is readily checked that Δ has an $\mathbf{L}_m^2$ gain less than or equal to γ if and only if $\tilde{\Delta}$ is passive. (This equivalence is shown, for example, in [1]).

Two cases are now possible. Either $\mathcal{G}$ is unstable, which means that system $\mathcal{L}$ is unstable when the perturbation $\Delta = \gamma I$. Otherwise, we can apply the passivity theorem [7] to system $\tilde{\mathcal{L}}$: System $\tilde{\mathcal{L}}$ is robustly stable if and only if $\mathcal{G}$ is strictly passive, which is equivalent (see for example [7]) to the frequency-domain condition that for some $\epsilon > 0$,

$$G(j\omega) + G(j\omega)^* \geq 2\epsilon I \text{ for all } \omega \in \mathbf{R}.$$

Suppose that in addition to being passive, $\tilde{\Delta}$ also satisfies other assumptions. Then the passivity theorem yields only a sufficient condition for robust stability. In this case, we can employ *multiplier theory* to utilize this additional information in order to obtain less conservative conditions for robust stability.

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[1] See [7, Section VI.11] for a discussion of this point.

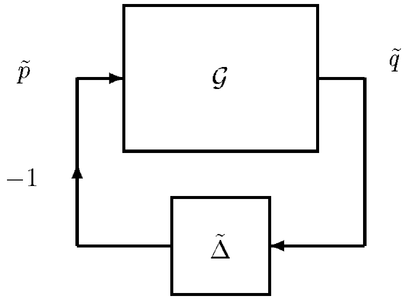

Figure 2: System $\tilde{\mathcal{L}}$: Passivity theorem framework

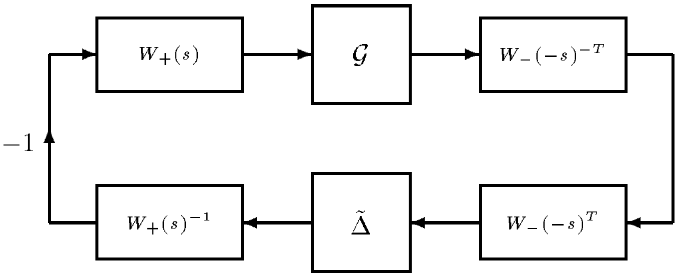

Figure 3: System with multipliers

Consider the system in Figure 3, where $W_+(s)$ and $W_-(s)$ are transfer matrices of appropriate sizes. Suppose $W_+(s)$ and $W_-(s)$ satisfy

$$W_+(s) \text{ and } W_-(-s) \text{ are stable and proper, with stable and proper inverses.} \tag{3}$$

Then, robust stability of system in Figure 3 is equivalent to robust stability of the system $\tilde{\mathcal{L}}$. Next, suppose that the operator

$$W_+(s)^{-1} \circ \tilde{\Delta} \circ W_-(-s)^T \text{ is passive.} \tag{4}$$

Then from the passivity theorem, system $\tilde{\mathcal{L}}$ is stable if

$$W_-(-s)^{-T} G(s) W_+(s) \text{ is strictly passive.} \tag{5}$$

Condition (5) is equivalent to the frequency domain criterion that for some $\epsilon > 0$,

$$W_-(j\omega)^{-*} G(j\omega) W_+(j\omega) + W_+(j\omega)^* G(j\omega)^* W_-(j\omega)^{-1} \geq 2\epsilon I \text{ for all } \omega \in \mathbf{R},$$

which in turn is equivalent to the condition that for some $\epsilon > 0$,

$$G(j\omega) W(j\omega) + W(j\omega)^* G(j\omega)^* \geq 2\epsilon I \text{ for all } \omega \in \mathbf{R}, \tag{6}$$

where $W \triangleq W_+ W_-$ is called the *stability multiplier*.

Thus, robust stability analysis using multipliers involves finding W such that:

(i) W can be factorized into W_+W_- where W_- and W_+ satisfy (3) and (4), and

(ii) W and G satisfy (6).

The additional information about Δ, and therefore $\tilde{\Delta}$, influences the choice of $W = W_+W_-$ through condition (4).

With the substitution $G(s) = (I - \gamma H(s))^{-1}(I + \gamma H(s))$, condition (6) is equivalent to the condition that for some $\epsilon > 0$,

$$W_{\mathrm{H}}(j\omega) - \gamma^2 H(j\omega)W_{\mathrm{H}}(j\omega)H(j\omega)^* + \gamma\left(H(j\omega)W_{\mathrm{S}}(j\omega) - W_{\mathrm{S}}(j\omega)H(j\omega)^*\right) \geq 2\epsilon I \text{ for all } \omega \in \mathbf{R}, \quad (7)$$

where $W_{\mathrm{H}}(s) = W(s) + W(-s)^T$ and $W_{\mathrm{S}}(s) = W(s) - W(-s)^T$.

With these preliminaries, we now consider how some standard robustness tests for system $\mathcal{L}$ can be performed using system $\tilde{\mathcal{L}}$.

3.1 Diagonal perturbations

Suppose that Δ, in addition to having an $\mathbf{L}_m^2$ gain less than γ, is also diagonal. By this, we mean that the operator Δ can be also specified by $p_i = \delta_i q_i$, where each single-input single-output nonlinear operator δ_i has an $\mathbf{L}^2$ gain less than γ. (The extension to the block-diagonal case, or when there are additional equality constraints such as $\delta_i = \delta_j$ is straightforward.) Note that $\tilde{\Delta}$ is diagonal as well.

Let W be any constant diagonal positive-definite matrix. Then W can be factored into W_+W_- with $W_+ > 0$ and $W_- > 0$, which satisfy (3) and (4) (for instance, we can take $W_+ = W$, $W_- = I$). Therefore, system $\mathcal{L}$ is robustly stable if for some diagonal $W > 0$, and some $\epsilon > 0$,

$$G(j\omega)W + WG(j\omega)^* \geq 2\epsilon I \text{ for all } \omega \in \mathbf{R}. \quad (8)$$

Correspondingly, condition (7) is that for some $\epsilon > 0$,

$$W - \gamma^2 H(j\omega)WH(j\omega)^* \geq 2\epsilon I \text{ for all } \omega \in \mathbf{R}. \quad (9)$$

Inequality (9) is the well-known condition that the scaled $\mathbf{H}_\infty$ norm of H, minimized over all diagonal constant scalings, be less than $1/\gamma$ (see for example [6]).

3.2 Diagonal LTI perturbations

Next, suppose that Δ is a diagonal LTI perturbation with $\mathbf{L}_m^2$ gain less than or equal to γ. Then $\tilde{\Delta}$ is diagonal and passive.

Suppose $W(s)$ is diagonal, bounded on the imaginary axis, and satisfies, for some $\epsilon > 0$,

$$W(j\omega) = W(j\omega)^* \geq 2\epsilon I \text{ for all } \omega \in \mathbf{R}. \quad (10)$$

Then W can be factorized into W_+ and W_- satisfying (3) (see [7, Section VI.9.4–5]). Moreover, the factors W_- and W_+ thus obtained satisfy condition (4). This can be seen as follows. Since $\tilde{\Delta}$ is an LTI perturbation, condition (4) is equivalent to the frequency-domain condition that

$$W_+(j\omega)^{-1}\tilde{\Delta}(j\omega)W_-(j\omega)^* + W_-(j\omega)\tilde{\Delta}(j\omega)^*W_+(j\omega)^{-*} \geq 0 \text{ for all } \omega \in \mathbf{R},$$

or equivalently

$$W_-(j\omega)^{-1}W_+(j\omega)^{-1}\tilde{\Delta}(j\omega) + \tilde{\Delta}(j\omega)^* W_+(j\omega)^{-*}W_-(j\omega)^{-*} \geq 0 \text{ for all } \omega \in \mathbf{R}. \tag{11}$$

The condition that $\tilde{\Delta}$ is passive means that the Nyquist plot of each diagonal entry of $\tilde{\Delta}$ lies in the closed right half complex plane. Since the product $W = W_+W_-$ satisfies, for some $\epsilon > 0$, condition (10), the Nyquist plot of $W^{-1}\tilde{\Delta}$ also lies in the closed right half complex plane, i.e., condition (11) is satisfied.

Let us next examine (6). In this case, we require, for some $\epsilon > 0$,

$$G(j\omega)W(j\omega) + W(j\omega)G(j\omega)^* \geq 2\epsilon I \text{ for all } \omega \in \mathbf{R}, \tag{12}$$

where W satisfies (10). Correspondingly, condition (7) is that for some $\epsilon > 0$,

$$W(j\omega) - \gamma^2 H(j\omega)W(j\omega)H(j\omega)^* \geq 2\epsilon I \text{ for all } \omega \in \mathbf{R}. \tag{13}$$

Inequality (13) is the well-known condition that the $\mathbf{H}_\infty$ norm of H, minimized over all diagonal *frequency-dependent scalings*, be less than $1/\gamma$ (see, for example, [6]).

3.3 Diagonal parametric perturbations

Next, suppose that Δ is a constant diagonal matrix with entries known to lie in $[-\gamma, \gamma]$. Then $\tilde{\Delta}$ is a constant diagonal matrix with entries in $[0, \infty)$.

Suppose $W(s)$ is diagonal, bounded on the imaginary axis, and satisfies, for some $\epsilon > 0$,

$$W(j\omega) + W(j\omega)^* \geq 2\epsilon I \text{ for all } \omega \in \mathbf{R}. \tag{14}$$

Then W can be factorized into W_+ and W_- satisfying (3). Moreover, the factors W_- and W_+ thus obtained satisfy condition (4). This can be seen as follows. Since $\tilde{\Delta}$ is a constant diagonal matrix with nonnegative diagonal entries, condition (4) is equivalent to the frequency-domain condition that

$$W_+(j\omega)^{-1}\tilde{\Delta}W_-(j\omega)^* + W_-(j\omega)\tilde{\Delta}W_+(j\omega)^{-*} \geq 0 \text{ for all } \omega \in \mathbf{R},$$

or equivalently

$$W_-(j\omega)^{-1}W_+(j\omega)^{-1}\tilde{\Delta} + \tilde{\Delta}W_+(j\omega)^{-*}W_-(j\omega)^{-*} \geq 0 \text{ for all } \omega \in \mathbf{R}. \tag{15}$$

In other words, we require the Nyquist plot of every diagonal entry of $\tilde{\Delta}W^{-1}$ to lie in the closed right half plane. Since $\tilde{\Delta}$ is a constant diagonal positive semi-definite matrix, we simply require every diagonal entry of W to have a Nyquist plot that lies in the closed right half plane; this is ensured by condition (14).

Let us next examine the condition (6). We then require, for some $\epsilon > 0$,

$$G(j\omega)W(j\omega) + W(j\omega)^* G(j\omega)^* \geq 2\epsilon I \text{ for all } \omega \in \mathbf{R}. \tag{16}$$

In this case inequality (7) is that for some $\epsilon > 0$,

$$W_{\mathrm{H}}(j\omega) - \gamma^2 H(j\omega) W_{\mathrm{H}}(j\omega) H(j\omega)^* + \gamma \left(H(j\omega) W_{\mathrm{S}}(j\omega) - W_{\mathrm{S}}(j\omega) H(j\omega)^* \right) \geq 2\epsilon I \text{ for all } \omega \in \mathbf{R}, \quad (17)$$

which is well-known condition condition due to Fan, Tits and Doyle [8] for robust stability with real parametric perturbations.

4 Numerical implementation of stability tests using LMIs

All the tests described in the previous section are of the form "Find a multiplier W satisfying certain constraints such that GW satisfies a frequency domain condition". We now show how these tests can be (numerically) implemented using convex optimization techniques.

The fundamental result that enables us to check the frequency domain conditions of the previous section is the following: A transfer matrix $\tilde{H}$ with state-space realization $(\tilde{A},\ \tilde{B},\ \tilde{C},\ \tilde{D})$ satisfies $\tilde{H}(j\omega) + \tilde{H}(j\omega)^* \geq 2\epsilon I$ for all $\omega \in \mathbf{R}$, if and only if there exists a symmetric matrix Q satisfying the matrix inequality

$$\begin{bmatrix} \tilde{A}Q + Q\tilde{A}^T & \tilde{B} - Q\tilde{C}^T \\ \tilde{B}^T - \tilde{C}Q & 2\epsilon I - (\tilde{D} + \tilde{D}^T) \end{bmatrix} \leq 0$$

(see for example, [12, 2]). This matrix inequality is affine in the variable Q, and is referred to as a linear matrix inequality (or LMI) in Q. The important point here is that an LMI is a convex constraint with a special structure, and consequently, there exist efficient algorithms for checking its feasibility. We refer the reader to [10, 4] for details.

Let $(A,\ B,\ C,\ D)$ be a state-space realization of $H(s)$. Then it is easy to verify that $G(s) = (I - \gamma H(s))^{-1}(I + \gamma H(s))$ has a state space realization $(A_G,\ B_G,\ C_G,\ D_G)$, where

$$A_G = A + \gamma B(I - \gamma D)^{-1} C,\ B_G = 2B(I - \gamma D)^{-1},\ C_G = (I - \gamma D)^{-1} \gamma C,\ D_G = (I + \gamma D)(I - \gamma D)^{-1}.$$

4.1 Diagonal perturbations

$W > 0$ is a constant diagonal matrix. Then GW has a state-space realization $(A_{GW},\ B_{GW},\ C_{GW},\ D_{GW})$ where

$$A_{GW} = A_G, \quad B_{GW} = B_G W, \quad C_{GW} = C_G, \quad D_{GW} = D_G W.$$

Therefore, condition (8) is equivalent to the LMI in $Q = Q^T$, diagonal $W > 0$ and $\epsilon > 0$:

$$\begin{bmatrix} A_G Q + Q A_G^T & B_G W - Q C_G^T \\ W B_G^T - C_G Q & 2\epsilon I - (D_G W + W D_G^T) \end{bmatrix} \leq 0. \quad (18)$$

4.2 Diagonal LTI perturbations

The problem here is to check if there exists W satisfying (10) so that condition (12) is satisfied. In order to perform this test numerically, we restrict W to lie in an affine set

$$\mathcal{W} \triangleq \left\{ W_0 + \sum_{i=1}^{m} \theta_i W_i \;\middle|\; \theta \in \mathbf{R}^m, \quad W_i(s) = W_i(-s)^T \right\}, \quad (19)$$

where W_i are fixed, real-rational, diagonal transfer matrices. Thus, every $W \in \mathcal{W}$ has a realization $(A_W,\ B_W(\theta),\ C_W,\ D_W(\theta)$ with B_W and D_W *affine* functions of θ. Therefore, condition (10) is equivalent to an LMI in $Q_W = Q_W^T$, θ and $\epsilon > 0$:

$$\begin{bmatrix} A_W Q_W + Q_W A_W^T & B_W(\theta) - Q_W C_W^T \\ B_W(\theta)^T - C_W Q_W & 2\epsilon I - (D_W(\theta) + D_W(\theta)^T) \end{bmatrix} \leq 0. \tag{20}$$

Next, GW has a state-space realization $(A_{GW},\ B_{GW},\ C_{GW},\ D_{GW})$ where

$$A_{GW} = \begin{bmatrix} A_W & 0 \\ B_G C_W & A_G \end{bmatrix}, \quad B_{GW}(\theta) = \begin{bmatrix} B_W(\theta) \\ B_G D_W(\theta) \end{bmatrix},$$
$$C_{GW} = [D_G C_W \quad C_G], \quad D_{GW}(\theta) = D_G D_W(\theta).$$

Note that B_{GW} and D_{GW} are affine functions of θ. Then checking condition (12) is equivalent to the LMI in variables $Q = Q^T$, θ and $\epsilon > 0$:

$$\begin{bmatrix} Q A_{GW}^T + A_{GW} Q & B_{GW}(\theta) - Q C_{GW}^T \\ B_{GW}(\theta)^T - C_{GW} Q & 2\epsilon I - \left(D_{GW}(\theta) + D_{GW}(\theta)^T\right) \end{bmatrix} \leq 0. \tag{21}$$

4.3 Diagonal parametric perturbations

The numerical procedure here is very similar to the one in the previous subsection. Here we restrict the multipliers W to lie in an affine set

$$\mathcal{W} \triangleq \left\{ W_0 + \sum_{i=1}^{m} \theta_i W_i \;\middle|\; \theta \in \mathbf{R}^m \right\}, \tag{22}$$

where W_i are fixed, real-rational, diagonal transfer matrices. Note that unlike with LTI perturbations, we do not require $W_i(s) = W_i(-s)^T$. Once again, condition (14) is equivalent to the LMI condition (20), and checking condition (16) is equivalent to the LMI (21).

5 An example

We take

$$H(s) = \begin{bmatrix} 2 & \dfrac{-10s - 8}{5(s+1)} \\ \dfrac{-2s + 8}{(s+1)} & 2 \end{bmatrix}.$$

5.1 Diagonal perturbations

We first consider the case when Δ is diagonal. We find, using LMI (18) and a simple bisection scheme, that the largest size γ of diagonal Δ against which the system is guaranteed to be stable (using our approach) is about 0.2082; the corresponding multiplier is $W = \mathbf{diag}(0.3867, 1.3454)$.

For a plot of the minimum eigenvalues of the Hermitian parts of $G(j\omega)$ and $G(j\omega)W$, we refer the reader to Figure 4 in [3].

5.2 Diagonal LTI perturbations

We next consider diagonal LTI Δ. The multipliers we employ are diagonal, with diagonal entries of the form $w_0 + w_1/(s^2-1)$. (This is an *ad hoc* choice.) Then, using LMIs (20) and (21) and a simple bisection scheme, we find that the largest size γ of diagonal LTI Δ against which the system is guaranteed to be stable (using our approach) is about 0.2217; the corresponding multiplier $W(s)$ is

$$\mathbf{diag}\left(1.3299 - \frac{0.8419}{(s^2-1)},\ 1.1755 - \frac{8.9447}{(s^2-1)}\right).$$

For a plot of the minimum eigenvalues of the Hermitian parts of $G(j\omega)$ and $G(j\omega)W(j\omega)$, we refer the reader to Figure 5 in [3].

5.3 Diagonal parametric perturbations

When Δ is a constant, unknown diagonal matrix, we use diagonal multipliers $W(s)$ with diagonal entries of the form $w_0 + w_1/(s+1) + w_2/(s+1)^2$ (again an *ad hoc* choice). Then, using LMIs (20) and (21) and a simple bisection scheme, we find that the largest size γ of diagonal parametric Δ against which the system is guaranteed to be stable (using our approach) is about 0.2414; the corresponding multiplier is

$$\mathbf{diag}\left(0.2961 + \frac{0.1206}{s+1} - \frac{0.0934}{(s+1)^2},\ 0.5021 + \frac{1.2464}{s+1} - \frac{0.1630}{(s+1)^2}\right).$$

For a plot of the minimum eigenvalues of the Hermitian parts of $G(j\omega)$ and $G(j\omega)W(j\omega)$, we refer the reader to Figure 6 in [3].

6 Conclusions

We have reinterpreted a number of standard robustness tests in the framework of the passivity theorem with multipliers. We have then shown how these tests can be performed using convex optimization over linear matrix inequalities. Several extensions to the results presented here are possible. The first is a systematic method for choosing the basis W_i for the multipliers in (19) and (22). The second is efficient computation of the largest γ for which we can guarantee robust stability using multipliers (recall that we have used a bisection scheme in this paper). This quantity has the interpretation of a lower bound on the robust stability margin [11] (and yields an upper bound on "μ" [5]). The third is feedback synthesis (or robust stabilization). Finally, the extension of the techniques presented in this paper to the computation of robust performance measures is also of interest.

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