Lyapunov Functionals in Complex-μ Analysis

V. Balakrishnan
School of Electrical and Computer Engineering
Purdue University
West Lafayette, IN 47907–1285
Internet: ragu@ecn.purdue.edu

Abstract

Conditions for robust stability of linear time-invariant systems subject to structured linear time-invariant uncertainties can be derived in the complex-μ framework, or equivalently in the framework of integral quadratic constraints. These conditions can be checked numerically with LMI-based convex optimization using the Kalman-Yakubovich-Popov Lemma. In this paper, we show how LMI tests also yield a *convex* parametrization of (a subset of) Lyapunov functionals that prove robust stability of such uncertain systems. We show that for uncertainties that are pure delays, the Lyapunov functionals reduce to the standard Lyapunov-Krasovksii functionals that are encountered in the stability analysis of delay systems. We demonstrate the practical utility of the Lyapunov functional parametrization by deriving bounds for a number of measures of robust performance (beyond the usual $\mathbf{H}_\infty$ performance); these bounds can be efficiently computed using convex optimization over linear matrix inequalities.

1 Introduction

Most control system models in use today incorporate in them explicitly "uncertainties" or "perturbations". These uncertainties may model a number of factors, including: *dynamics that are neglected* to make the model tractable, as with large scale structures; *nonlinearities* that are either hard to model or too complicated; *parameters* that are not known exactly, either because they are hard to measure or because of varying manufacturing conditions. Robust control deals with the analysis of and design for such control system models; see, for example, [1, 2].

Perhaps the most popular paradigm currently in use for robust control has a nominal finite-dimensional, linear, time-invariant system $\mathcal{H}$ with the perturbation or uncertainty Δ in the feedback loop (see Figure 1). The uncertainty is assumed to be bounded in size; typically this is stated as a norm constraint on Δ. Often additional information about Δ is either known or assumed: diagonal or block-diagonal; sector-bounded memoryless, linear time-invariant or parametric, etc. In such cases, the perturbations are called "structured perturbations". In this paper, we restrict our attention to the case when Δ is block-diagonal with each block being a linear time-invariant operator (LTI uncertainty). This is known popularly as the complex-μ framework.

One of the most fundamental questions concerning the system in Figure 1 is that of stability: "Is the model stable irrespective of the perturbation Δ?" This is also referred to as the robust stability problem. The best-known early approaches towards solving this problem were the μ-analysis methods due to Doyle [3], and the K_m analysis methods due to Safonov [4]. These approaches derive a necessary and sufficient condition for robust stability that the function "μ" of the frequency response of $\mathcal{H}$ must not exceed one at any frequency. One way to (approximately) check robust stability is to verify, at a finite number of frequencies, that an upper bound for the μ value of the frequency response does not exceed one [5, 1]; this in effect serves as a sufficient condition for robust stability. Efficient implementations of such techniques are available; see [6].

More recently, it has been observed that stability conditions stated in terms of the upper bounds of complex-μ can be re-derived in a unified setting using multiplier theory [7, 8, 9], or equivalently in the framework of integral quadratic constraints (IQCs) [10]. The advantage afforded by this observation is that the sufficient conditions for robust stability can be now checked without frequency sampling, using convex optimization techniques based on linear matrix inequalities [9, 11, 10, 12].

In this paper, we demonstrate another advantage in using the IQC approach. We show how the LMI implementation of robust stability tests for a system with LTI uncertainties yields a *convex* parametrization of (a subset of) Lyapunov functionals that prove robust stability of the uncertain system. Consequently, one may optimize over the set of Lyapunov functionals numerically very easily in order to derive guaranteed bounds on measures of robust performance, including not only the usual "$\mathbf{H}_\infty$ performance" (i.e., the worst-case $\mathbf{H}_\infty$ norm of a closed-loop transfer function of interest), but also quantities such as the "$\mathbf{H}_2$ performance" (i.e., the worst-case $\mathbf{H}_2$ norm of a closed-loop transfer function of interest), and bounds on the state variables. We illustrate this with a number of such performance measures.

In principle, the derivation of Lyapunov functions from μ or multiplier analysis is straightforward: In case of μ analysis, first obtain a rational function that interpolates the scalings; in the case of multiplier analysis, such a rational function is immediately available. Then perform a spectral factorization of this rational function. A Lyapunov function that proves the stability of the closed-loop system can be then constructed from the spectral factors. While a convex parametrization of the set of scalings or multipliers can be given, there is no analogous convex parametrization of the spectral factors; this renders the direct approach for constructing Lyapunov functions of not much practical use.

The Kalman-Yakubovich-Popov Lemma offers promise for a convex parametrization of Lyapunov functions. Using this Lemma, the frequency-domain condition that a finite-dimensional multiplier proves robust stability of the uncertain system can be rewritten as a linear matrix inequality in a symmetric matrix P; this matrix P, which we will call a "KYP certificate" plays a central role in the construction of any Lyapunov functions or functionals. Unfortunately, a positive-definite P may not always exist; indeed, precise conditions under which there exists a positive-definite KYP certificate are yet unknown [13, 14].

Our main contributions are as follows. First, we consider the question of when positive-definite KYP certificates exist. We show that given a multiplier that proves robust stability of the uncertain system, we can always find a realization for the multiplier such that positive-definite KYP certificates exist. Equivalently, we can always translate the set of KYP certificates for a frequency-domain inequality so that their positive-definiteness can be assured. We devote §3 to these observations, as we believe them to be of independent interest. We then proceed towards deriving a convex parametrization of Lyapunov functionals for μ analysis, in §4. This parametrization requires no spectral factorization, is described by simple linear matrix inequalities, and yields as a special case the well-known Lyapunov-Krasovksii functionals that are encountered in the stability analysis of LTI systems with delays. We demonstrate the utility of this parametrization in §5, where we derive upper bounds for several measures of performance for systems with LTI uncertainties that can be computed using convex optimization over linear matrix inequalities. We also present a simple numerical example in §5.

Notation

The notation and terminology are standard; for details and precise technical conditions, we refer the reader to the book by Desoer and Vidyasagar [15].

$\mathbf{R}$ ($\mathbf{R}_+$) denotes the set of real (nonnegative real) numbers. $\mathbf{L}_2$ is the Hilbert space of square-integrable signals defined over $\mathbf{R}_+$. $\mathbf{L}_\infty$ is space of bounded signals defined over $\mathbf{R}_+$. $\mathbf{L}_{2e}$ denotes the extended space associated with $\mathbf{L}_2$. The $\mathbf{H}_\infty$ norm of the transfer matrix H of a causal LTI system $\mathcal{H}$ is denoted $\|H\|_\infty$, and defined as $\|H\|_\infty = \sup_{\mathrm{Re}\, s>0} \sigma_{\max}(H(s))$, where $\sigma_{\max}(M)$ is the maximum singular value of the matrix M. Of course, $\mathcal{H}$ is stable if and only if $\|H\|_\infty$ is finite, in which case $\|H\|_\infty$ equals the $\mathbf{L}_2$ gain of $\mathcal{H}$. For a strictly proper LTI system with transfer function $H(s)$ and impulse response $h(t)$, the $\mathbf{H}_2$ norm is denoted $\|H\|_2$, and defined as

$$\|H\|_2^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{Tr}\ H(j\omega)^* H(j\omega)\, d\omega = \int_0^\infty \mathrm{Tr}\ h(t)^T h(t)\, dt,$$

where $\mathbf{Tr}\ (M)$ denotes the trace of a square matrix M.

The matrix inequalities $A > B$ and $A \geq B$ mean A and B are square, Hermitian, and that $A-B$ is positive definite and positive semi-definite, respectively. $A^{1/2}$ denotes the Hermitian square-root of $A \geq 0$.

2 Preliminaries

We consider a standard paradigm for robust control, shown in Figure 1, consisting of a stable finite-dimensional linear system $\mathcal{H}$ with an uncertainty or perturbation $\mathit{\Delta}$ in the feedback loop:

$$\frac{d}{dt}x(t) = Ax(t) + B_p p(t), \quad q(t) = C_q x(t) + D_{qp} p(t), \quad p(t) = \Delta(q,t), \tag{1}$$

where $A \in \mathbf{R}^{n\times n}$, $B_p \in \mathbf{R}^{n\times n_p}$, $C_q \in \mathbf{R}^{n_q\times n}$, and $D_{qp} \in \mathbf{R}^{n_q\times n_p}$, and (A, B_p, C_q, D_{qp}) is a minimal state-space realization of the linear system $\mathcal{H}$.

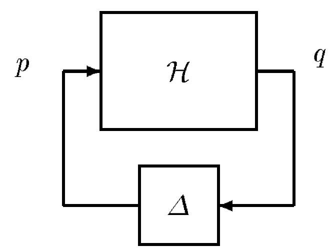


Figure 1: A standard framework for robustness analysis

Throughout this paper, we assume that $n_p = n_q = m$; with little difficulty, the results herein can be extended to the more general case $n_p \neq n_q$. For future reference, we note that the transfer function of the linear system $\mathcal{H}$ is given by $H(s) = C_q(sI - A)^{-1}B_p + D_{qp}$. We assume that the perturbation $\mathit{\Delta}$ is LTI with a transfer function Δ. Henceforth we drop the distinction between the perturbation $\mathit{\Delta}$ and its transfer function Δ, and with some abuse of notation, refer to the perturbation by its transfer function. We make two further assumptions regarding Δ:

1. We assume that Δ is stable and causal, with an $\mathbf{L}_2$ gain that does not exceed one. Thus
$$\|\Delta\|_\infty \leq 1. \tag{2}$$

2. We assume that Δ is a structured LTI uncertainty. More precisely,
$$\begin{gathered}\Delta(s) = \mathrm{diag}\left(\delta_1^c(s)I_{k_1}, \ldots, \delta_{m_c}^c(s)I_{k_{m_c}}, \Delta_1^C(s), \ldots, \Delta_{m_C}^C(s)\right), \\ \text{with } \delta_q^c : \mathbf{C} \to \mathbf{C}, \Delta_q^C : \mathbf{C} \to \mathbf{C}^{k_{m_c+q}\times k_{m_c+q}}.\end{gathered} \tag{3}$$

Though the notation in (3) appears rather complicated, the underlying idea is quite simple: Δ is block-diagonal, with the first m_c blocks representing single-input single-output LTI uncertainties with the qth uncertainty repeated k_q times, and the next m_C blocks representing multi-input multi-output LTI uncertainties with the qth uncertainty having k_{m_c+q} inputs and k_{m_c+q} outputs.

We let $\mathbf{\Delta}$ denote the set of all Δ satisfying (2) and (3).

Remark 1 The control system model given by (1) where $\Delta \in \mathbf{\Delta}$ has been found to be of considerable practical relevance; see for example [1]. We point out that linear systems with LTI uncertainties that enter polynomially into the transfer function coefficients can also be cast in the above framework; see for instance [16]. $\diamond$

Equations (1) and $\mathbf{\Delta}$ together may be thought of as representing a set of linear time-invariant systems, each member corresponding to some $\Delta \in \mathbf{\Delta}$. We say that system (1) is "robustly stable over $\mathbf{\Delta}$" if (1) is stable for all $\Delta \in \mathbf{\Delta}$. Our main objective is to derive a convex parametrization of a set of Lyapunov functionals that prove robust stability of system (1) over $\mathbf{\Delta}$.

We now state a sufficient condition for robust stability of system (1) over $\mathbf{\Delta}$. This condition is known as the complex-μ stability condition [1]. Let

$$\mathcal{W} = \left\{ W \in \mathbf{C}^{m\times m} : \begin{array}{l} W = \mathrm{diag}(W_1^c, \ldots, W_{m_c}^c, w_1^C I_{k_{m_c+1}}, \ldots, w_{m_C}^C I_{k_{m_c+m_C}}), \\ W_i^c = (W_i^c)^* \in \mathbf{C}^{k_i \times k_i}, i = 1, \ldots, m_c, \\ w_i^C \in \mathbf{R}, i = 1, \ldots, m_C \end{array} \right\}, \tag{4}$$

and

$$\mathbf{\Pi} \stackrel{\Delta}{=} \{\Pi : j\mathbf{R} \to \mathcal{W}, \text{ and for some } \epsilon > 0,\ \Pi(j\omega) \geq 2\epsilon I \text{ for all } \omega \in \mathbf{R}\}. \tag{5}$$

Remark 2 The set $\mathcal{W}$ that is fundamental to the definition of the set $\mathbf{\Pi}$ has the property that its every element commutes with every $\Delta \in \mathbf{\Delta}$. $\diamond$

Note that for every $\Delta \in \mathbf{\Delta}$ and every $\Pi \in \mathbf{\Pi}$, we have

$$\Delta(j\omega)^* \Pi(j\omega) \Delta(j\omega) - \Pi(j\omega) \leq 0 \quad \text{for all } \omega \in \mathbf{R}. \tag{6}$$

The set $\mathbf{\Pi}$ is often referred to as the set of stability multipliers, and can be regarded as representing the "knowledge" of the structure and nature of the uncertainties. The characterization of $\mathbf{\Delta}$ via (6) can be regarded a special case of the more general characterization of uncertainties via integral quadratic constraints (IQCs) [10].

We then have the following sufficient condition for robust stability of system (1) [3, 4, 10].

Theorem 1 *Suppose that there exist* $\Pi \in \mathbf{\Pi}$ *and* $\epsilon > 0$ *such that for all* $\omega \in \mathbf{R}$,

$$H(j\omega)^* \Pi(j\omega) H(j\omega) - \Pi(j\omega) \leq -2\epsilon I. \tag{7}$$

Then, system (1) is robustly stable over $\mathbf{\Delta}$.

3 Frequency-domain inequalities and the KYP Lemma

The Kalman-Yakubovich-Popov Lemma plays a key role in the numerical verification of frequency-domain inequalities such as (7) via convex optimization based on linear matrix inequalities. For convenience, we define

$$F(A, B, M, s) = \begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix}.$$

Then we have the following lemma[1].

Lemma 1 (KYP Lemma) *Let* $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$ *and* $M = M^T \in \mathbf{R}^{(m+n)\times(m+n)}$, *with* A *having no eigenvalues on the imaginary axis. Then, the following statements are equivalent.*

1. *For some* $\epsilon > 0$,

$$F(A, B, M, j\omega) \geq 2\epsilon I, \quad \text{for all } \omega \in \mathbf{R}. \tag{8}$$

2. *There exists* $P = P^T$ *such that*

$$\begin{bmatrix} A^TP + PA & PB \\ B^TP & 0 \end{bmatrix} < M. \tag{9}$$

We say that $F(A, B, M, \cdot\,)$ is positive on the imaginary axis if condition (8) holds. Owing to the equivalence between (8) and (9), we say that the matrix P is a KYP certificate that proves the positivity of $F(A, B, M, \cdot\,)$ on the imaginary axis.

Remark 3 The matrix inequality (9) is affine in the variable P, and is therefore a linear matrix inequality or LMI in P [17]. The set of P satisfying (9) is a convex set. Checking whether or not this set is empty can be performed very efficiently using numerical methods [17, 18, 19]. $\diamond$

Remark 4 Suppose that the matrix inequality (9) is feasible. Then, along each trajectory x of the linear system $\dot{x} = Ax + Bp$, the functional $V : \mathbf{L}_{2e} \times \mathbf{R}_+ \to \mathbf{R}$ given by

$$V(x, t) = x(t)^T P x(t) - \int_0^t \begin{bmatrix} x(\tau) \\ p(\tau) \end{bmatrix}^T M \begin{bmatrix} x(\tau) \\ p(\tau) \end{bmatrix} d\tau$$

[1]The version of the KYP Lemma quoted here is taken from [14].

satisfies $\frac{d}{dt}V(x,t) \leq 0$. If P is positive-definite, then the first term of $V(x,t)$ is nonnegative for all x and t. If the second-term of $V(x,t)$ also happens to be nonnegative for all x, p and t, we may regard the matrix P as providing a recipe for the construction a Lyapunov functional $V(x,t)$, for every p, that proves the boundedness of x. Such an argument will play an important role in §4.

However, the matrix P in LMI (9) is not necessarily positive-definite, even when A is Hurwitz. Indeed, as the following counterexample shows, every element of the feasible set of (9) can be negative definite: Let

$$A = -1, \quad B = 1, \quad \text{and} \quad M = \begin{bmatrix} 3.5 & -2 \\ -2 & 1 \end{bmatrix}. \tag{10}$$

Then, LMI (9) is equivalent to

$$(P+1)^2 - 0.5 < 0.$$

Thus for every P satisfying (9), we have $P < 0$. $\diamond$

Remark 5 It is also false that there exists $P = P^T > 0$ satisfying LMI (9) if the following stronger condition holds: For some $\epsilon > 0$,

$$F(A,B,M,s) \geq 2\epsilon I, \qquad \text{for all } s \text{ with } \Re s \geq 0. \tag{11}$$

This is demonstrated by the counterexample[2]

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}; \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad M = \begin{bmatrix} 0 & -2 & 1 \\ -2 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix} + \mu I, \tag{12}$$

where μ is small and positive. It is readily verified that while for some $\epsilon > 0$ condition (11) holds for all s with $\Re s \geq 0$ and $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ satisfies LMI (9), there is no $P = P^T > 0$ satisfying (9). $\diamond$

Thus, there remains the important question of necessary and sufficient conditions under which there exist positive-definite KYP certificates proving the positivity of $F(A,B,M,\cdot)$ on the imaginary axis. A partial answer to this question is provided by the following lemma.

Lemma 2 *Suppose (8) holds, and let $\mathcal{P}$ denote the set of KYP certificates that prove the positivity of $F(A,B,M,\,\cdot\,)$ on the imaginary axis. Define*

$$\tilde{M} \triangleq M - \begin{bmatrix} A^TX + XA & XB \\ B^TX & 0 \end{bmatrix}, \tag{13}$$

where $X = X^T \in \mathbf{R}^{n\times n}$. Then:

[2]The existence of counterexamples such as the one listed in this and the previous remark are well-known to researchers; see for example [13]. The ones here were communicated by Prof. Anders Rantzer by way of Dr. Fan Wang.

1. *$F(A, B, \tilde{M}, j\omega) = F(A, B, M, j\omega)$ for all $\omega \in \mathbf{R}$.*

2. *The set $\tilde{\mathcal{P}}$ of KYP certificates proving the positivity of $F(A, B, \tilde{M}, \cdot)$ on the imaginary axis is given by*
$$\tilde{\mathcal{P}} = \{P - X \ : \ P \in \mathcal{P}\}.$$

Proof:

1. Let $g(\omega) = \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}$. Then, for any $X = X^T \in \mathbf{R}^{n\times n}$, we have
$$\begin{aligned} g(\omega)^* \begin{bmatrix} A^TX + XA & XB \\ B^TX & 0 \end{bmatrix} g(\omega) &= g(\omega)^* \begin{bmatrix} -(j\omega I - A)^*X - X(j\omega I - A) & XB \\ B^TX & 0 \end{bmatrix} g(\omega) \\ &= 0. \end{aligned}$$
Then $F(A, B, \tilde{M}, j\omega) = F(A, B, M, j\omega)$ follows immediately.

2. Follows by direct verification.

$\square$

Remark 6 Lemma 2 shows that given $G : j\mathbf{R} \to \mathbf{C}^{m\times m}$ of the form $G(j\omega) = F(A, B, M, j\omega)$, we may associate with it a set of functions
$$\mathcal{F}(A, B, M) = \left\{ \tilde{F} : \tilde{F}(s) = F(A, B, \tilde{M}, s), \ \tilde{M} = M - \begin{bmatrix} A^TX + XA & XB \\ B^TX & 0 \end{bmatrix}, \ X = X^T \in \mathbf{R}^{n\times n} \right\}.$$
Every $\tilde{F} \in \mathcal{F}(A, B, M)$ matches G on the imaginary axis, with the parameter $X = X^T$ serving to "translate" its set of KYP certificates. As X is essentially a free parameter, by writing G as $G(j\omega) = F(A, B, \tilde{M}, j\omega)$ for some appropriate $\tilde{M}$, one can always find a positive-definite KYP certificate proving its positivity on the imaginary axis. $\diamond$

Remark 7 The comments made in Remark 6 can be interpreted in terms of coordinate transformations. The linear system whose transfer function is
$$G(s) = \begin{bmatrix} (-sI - A)^{-1}B \\ I \end{bmatrix}^T M \begin{bmatrix} (sI - A)^{-1}B \\ I \end{bmatrix} \tag{14}$$
has a state-space realization
$$\left(\begin{bmatrix} A & 0 \\ -M_{11} & -A^T \end{bmatrix}, \begin{bmatrix} B \\ -M_{12} \end{bmatrix}, \begin{bmatrix} M_{12}^T & B^T \end{bmatrix}, M_{22} \right), \tag{15}$$

where we have partitioned M in an obvious manner as $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{bmatrix}$. Replacing M in (14) by $\tilde{M}$ that satisfies (13) for some $X = X^T$ is equivalent to applying a change of coordinates to the realization (15) with transformation matrix

$$T = \begin{bmatrix} I & 0 \\ -X & I \end{bmatrix}.$$

$\diamond$

Remark 8 Given $G(j\omega) = F(A, B, M, j\omega)$ that is positive on the imaginary axis, with all eigenvalues of A having negative real part, one representation is of particular interest. It is easily argued that M_{22} is invertible. Then, let X_{m} be the solution of the Riccati equation

$$A^T X_{\mathrm{m}} + X_{\mathrm{m}} A - M_{11} + (X_{\mathrm{m}} B - M_{12}) M_{22}^{-1} (X_{\mathrm{m}} B - M_{12})^T = 0 \tag{16}$$

with the following properties:

- $X_{\mathrm{m}} = X_{\mathrm{m}}^T$.
- The column space of $\begin{bmatrix} I \\ X_{\mathrm{m}} \end{bmatrix}$ spans the invariant subspace of the Hamiltonian matrix

  $$\begin{bmatrix} A - BM_{22}^{-1} M_{12}^T & BM_{22}^{-1} B^T \\ M_{11} - M_{12} M_{22}^{-1} M_{12}^T & -(A - BM_{22}^{-1} M_{12}^T)^T \end{bmatrix}$$

  corresponding to its eigenvalues with negative real part.
- All eigenvalues of $A - BM_{22}^{-1} M_{12}^T + BM_{22}^{-1} B^T X_{\mathrm{m}}$ have negative real part.

(We refer to X_{m} as the *stabilizing solution* to (16).) Then,

$$\tilde{M} = M - \begin{bmatrix} A^T X_{\mathrm{m}} + X_{\mathrm{m}} A & X_{\mathrm{m}} B \\ B^T X_{\mathrm{m}} & 0 \end{bmatrix} = \begin{bmatrix} C & D \end{bmatrix}^T \begin{bmatrix} C & D \end{bmatrix},$$

where $C = -M_{22}^{-1/2} (B^T X_{\mathrm{m}} - M_{12}^T) \in \mathbf{R}^{m \times n}$ and $D = M_{22}^{1/2} \in \mathbf{R}^{m \times m}$. Moreover, $F(A, B, \tilde{M}, j\omega) = (C(j\omega I - A)^{-1} B + D)^* (C(j\omega I - A)^{-1} B + D)$ is a *spectral factorization* of $F(A, B, \tilde{M}, j\omega)$, that is $(C(j\omega I - A)^{-1} B + D)$ has a stable inverse. (See, for example, [20].) $\diamond$

4 Lyapunov functionals for complex-μ analysis

We now describe the numerical implementation of the stability test in Theorem 1 using the KYP Lemma. The set $\mathbf{\Pi}$ is not described by a finite number of variables. In order to reduce the number of optimization variables to a finite number, a standard technique is to define a subset of $\mathbf{\Pi}$ as follows. Let $\Psi_i : \mathbf{C} \to \mathbf{C}^{m\times m}$, $i = 0, \ldots, N$ be given by

$$\Psi_i(j\omega) = \frac{1}{(j\omega - \lambda)^i} I, \tag{17}$$

where $\lambda < 0$ is some real number, and I is the $m \times m$ identity matrix. (Note that $\Psi_0(jw) = I$ for all ω.)

Let

$$\Psi(j\omega) = \left[\begin{array}{ccccc} \Psi_0(j\omega)^* & \Psi_1(j\omega)^* & \Psi_2(j\omega)^* & \cdots & \Psi_N(j\omega)^* \end{array} \right]^*, \tag{18}$$

and let

$$\mathcal{M} = \left\{ M \in \mathbf{R}^{m(N+1)\times m(N+1)} : M = \begin{bmatrix} M_{00} & M_{01} & \ldots & M_{0N} \\ M_{10} & M_{11} & \ldots & M_{1N} \\ \vdots & \vdots & \ddots & \vdots \\ M_{N0} & M_{N1} & \ldots & M_{NN} \end{bmatrix}, \begin{array}{l} M_{ij} \in \mathcal{W} \cap \mathbf{R}^{m\times m}, \\ i = 0, \ldots, N, \\ j = 0, \ldots, N \end{array} \right\}. \tag{19}$$

Let

$$\mathbf{\Pi}_{\text{fin}} = \left\{ \Pi \ : \begin{array}{c} \Pi(j\omega) = \Psi(j\omega)^* R \Psi(j\omega), \ R = R^T \in \mathcal{M}, \\ \text{for some } \epsilon > 0, \ \text{for all } \omega \in \mathbf{R}, \quad \Psi(j\omega)^* R \Psi(j\omega) \geq 2\epsilon I \end{array} \right\}. \tag{20}$$

Clearly $\mathbf{\Pi}_{\text{fin}} \subset \mathbf{\Pi}$, and the idea is to search for $\Pi \in \mathbf{\Pi}_{\text{fin}}$ such that condition (7) holds.

Let $(A_\psi, B_\psi, C_\psi, D_\psi)$ be a minimal state-space realization of the system with transfer function $\Psi(j\omega)$, where $A_\psi \in \mathbf{R}^{n_v\times n_v}$, $B_\psi \in \mathbf{R}^{n_v\times m}$, $C_\psi \in \mathbf{R}^{m(N+1)\times n_v}$, $D_\psi \in \mathbf{R}^{m(N+1)\times m}$. Let

$$\tilde{A} = \begin{bmatrix} A & 0 & 0 \\ B_\psi C_q & A_\psi & 0 \\ 0 & 0 & A_\psi \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_p \\ B_\psi D_{qp} \\ B_\psi \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} D_\psi C_q & C_\psi & 0 \\ 0 & 0 & C_\psi \end{bmatrix}, \quad \tilde{D} = \begin{bmatrix} D_\psi D_{qp} \\ D_\psi \end{bmatrix}. \tag{21}$$

It is then easy to verify that the condition that inequality (7) holds for some Π with $\Pi(j\omega) = \Psi(j\omega)^* R \Psi(j\omega)$ can be rewritten as

$$F(\tilde{A}, \tilde{B}, \tilde{M}, j\omega) \leq -2\epsilon I, \tag{22}$$

where

$$\tilde{M} = \begin{bmatrix} \tilde{C}^T \\ \tilde{D}^T \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix} \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix}. \tag{23}$$

Then, the following lemma follows directly from Lemma 1. (See for example [11, 12].)

Lemma 3 *The following statements are equivalent.*

1. *There exists some* $\Pi \in \mathbf{\Pi}_{\text{fin}}$ *and* $\epsilon > 0$ *such that inequality (7) holds.*

2. *There exist matrices $P = P^T \in \mathbf{R}^{(n+2n_v)\times(n+2n_v)}$ and $X = X^T \in \mathbf{R}^{n_v\times n_v}$, and $R = R^T \in \mathcal{M}$, such that*

$$\begin{bmatrix} A_\psi^T X + XA_\psi & XB_\psi \\ B_\psi^T X & 0 \end{bmatrix} - \begin{bmatrix} C_\psi^T \\ D_\psi^T \end{bmatrix} R \begin{bmatrix} C_\psi & D_\psi \end{bmatrix} < 0, \tag{24a}$$

$$\begin{bmatrix} \tilde{A}^T P + P\tilde{A} & P\tilde{B} \\ \tilde{B}^T P & 0 \end{bmatrix} + \begin{bmatrix} \tilde{C}^T \\ \tilde{D}^T \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix} \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix} < 0. \tag{24b}$$

Remark 9 The idea underlying the parametrization of $\mathbf{\Pi}_{\text{fin}}$ as in (20) is that we may approximate a stable transfer function $G(s)$ as the truncated Laurent series $G(s) \approx \sum_{i=0}^{N} g_i(s-\lambda)^{-i}$, for some $\lambda < 0$. Restricting Π to lie in $\mathbf{\Pi}_{\text{fin}} \subseteq \mathbf{\Pi}$ using such a technique renders the numerical implementation of the stability test in Theorem 1 more conservative, i.e., it is a stronger requirement than is required by the theorem. The particular choice of $\Psi(j\omega)$ plays an important role in determining how much more conservatism the restriction $\Pi \in \mathbf{\Pi}_{\text{fin}}$ entails. The numerical implications of this issue are not fully understood; however, it can be shown [21] that the actual choices of λ and N in the definition of $\mathbf{\Pi}_{\text{fin}}$ are immaterial, provided N is large enough. $\diamond$

Remark 10 The case $N = 0$ above corresponds to the well-studied case of constant scalings. In this case, there are no difficulties with parametrizing positive-definite KYP certificates; indeed quadratic Lyapunov functions (rather than functionals) can be readily derived in this case; see for example [17]. $\diamond$

Note that as discussed in Remark 4, the KYP certificate P that proves (7) is not guaranteed to be positive-definite. We now state a modified version of Lemma 3 that guarantees the existence of positive-definite KYP certificates that prove (7).

Lemma 4 *The following statements are equivalent.*

1. *There exists some $\Pi \in \mathbf{\Pi}_{\text{fin}}$ and $\epsilon > 0$ such that inequality (7) holds.*

2. *There exists $R = R^T \in \mathcal{M}$ such that $R > 0$, and with $\Pi(j\omega) = \Psi(j\omega)^* R \Psi(j\omega)$, for some $\epsilon > 0$, inequality (7) holds.*

3. *There exist matrices $P = P^T \in \mathbf{R}^{(n+2n_v)\times(n+2n_v)}$ and $R = R^T \in \mathcal{M}$, such that*

$$P > 0, \quad R > 0, \quad \begin{bmatrix} \tilde{A}^T P + P\tilde{A} & P\tilde{B} \\ \tilde{B}^T P & 0 \end{bmatrix} + \tilde{M} < 0, \tag{25}$$

where $\tilde{A}$, $\tilde{B}$ and $\tilde{M}$ are defined in (21) and (23).

Proof: In Appendix A we show that we may, without loss of generality, enforce the condition that the variable R in the statement of Lemma 3 is positive-definite, and that as a consequence, we can guarantee the existence of a positive-definite P. The proof relies on the results of §3.

$\square$

The following theorem follows immediately.

Theorem 2 *Let $\tilde{A}$, $\tilde{B}$ and $\tilde{M}$ be as defined in (21) and (23). Suppose that there exist matrices $P = P^T \in \mathbf{R}^{(n+2n_v)\times(n+2n_v)}$ and $R = R^T \in \mathcal{M}$ such that LMI (25) holds. Then, system (1) is robustly stable over $\mathbf{\Delta}$.*

We next show how Theorem 2 can be used to provide a convex parametrization of a subset of Lyapunov functionals that prove robust stability of LTI systems with structured LTI uncertainties. This is the central result of the paper. While they are of independent interest, we will also see that these Lyapunov functionals can be used for performance analysis.

Theorem 3 *Suppose there exist $P = P^T$ and $R = R^T \in \mathcal{M}$ such that condition (25) holds. Consider any $\Delta \in \mathbf{\Delta}$, with its state initialized to zero. For some initial condition $x(0)$, let x, p and q be the solution to equations (1). Define*

$$x_1(t) = \int_0^t e^{A_\psi \nu} B_\psi q(t-\nu)\, d\nu, \quad x_2(t) = \int_0^t e^{A_\psi \nu} B_\psi p(t-\nu)\, d\nu. \tag{26}$$

Then, the function $V_\Delta : \mathbf{L}_{2e} \times \mathbf{R}_+ \to \mathbf{R}$, given by

$$V_\Delta(x,t) \triangleq \begin{bmatrix} x(t) \\ x_1(t) \\ x_2(t) \end{bmatrix}^T P \begin{bmatrix} x(t) \\ x_1(t) \\ x_2(t) \end{bmatrix} + \int_0^t \begin{bmatrix} x(\tau) \\ x_1(\tau) \\ x_2(\tau) \\ p(\tau) \end{bmatrix}^T \begin{bmatrix} \tilde{C}^T \\ \tilde{D}^T \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix} \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} x(\tau) \\ x_1(\tau) \\ x_2(\tau) \\ p(\tau) \end{bmatrix} d\tau \tag{27}$$

satisfies:

(a) *For some $\epsilon > 0$, $V_\Delta(x,t) \geq \epsilon x(t)^T x(t)$.* (28)

(b) *For some $\epsilon > 0$, $\frac{d}{dt} V_\Delta(x,t) \leq -\epsilon \left(x(t)^T x(t) + p(t)^T p(t) \right)$.* (29)

Proof: Consider any $\Delta \in \mathbf{\Delta}$ with its state initialized to zero.

Consider statement (a). Since $P > 0$, we have, for some $\epsilon > 0$,

$$\begin{bmatrix} x(t) \\ x_1(t) \\ x_2(t) \end{bmatrix}^T P \begin{bmatrix} x(t) \\ x_1(t) \\ x_2(t) \end{bmatrix} \geq \epsilon x(t)^T x(t).$$

We show in Appendix B that the second term in the definition of $V_\Delta(x,t)$ given in (27) is nonnegative, and therefore (a) follows.

Consider statement (b).

$$\frac{d}{dt}V_\Delta(x,t) = \begin{bmatrix} x(t) \\ x_1(t) \\ x_2(t) \\ p(t) \end{bmatrix}^T \begin{bmatrix} \tilde{A}^T P + P\tilde{A} & P\tilde{B} \\ \tilde{B}^T P & 0 \end{bmatrix} \begin{bmatrix} x(t) \\ x_1(t) \\ x_2(t) \\ p(t) \end{bmatrix} + \begin{bmatrix} x(t) \\ x_1(t) \\ x_2(t) \\ p(t) \end{bmatrix}^T \begin{bmatrix} \tilde{C}^T \\ \tilde{D}^T \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix} \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix} \begin{bmatrix} x(t) \\ x_1(t) \\ x_2(t) \\ p(t) \end{bmatrix}.$$

From (25) and the above equation, it follows immediately that for some $\epsilon > 0$,

$$\frac{d}{dt}V_\Delta(x,t) \leq -\epsilon\left(x(t)^T x(t) + p(t)^T p(t)\right).$$

□

Remark 11 For every $\Delta \in \mathbf{\Delta}$ with its state initialized to zero, $V_\Delta(x,t)$ is a *functional* of the state-trajectory x. This is to be contrasted with the more commonly-encountered Lyapunov *functions* for robust stability analysis and design; see, for example, [17, 22, 23, 24]. ◇

Remark 12 Note that it is *not true* that $V_\Delta(x,t) \to 0$ as $t \to \infty$. Theorem 3 implies only that for every $\Delta \in \mathbf{\Delta}$, the functional $V_\Delta(x,t)$ is a monotonically non-increasing function of time for every trajectory x, This immediately establishes that $x \in \mathbf{L}_\infty$, that is, $x(t)$ is bounded. (This is the so-called Lagrange stability property; see for example, [25].) Monotonicity of $V_\Delta(x,t)$ will prove crucial in deriving bounds on robust performance measures, in §5.

It can be shown that (28) and (29) also imply that system (1) is $\mathbf{L}_2$-stable: We have

$$0 \leq V_\Delta(x,t) = V_\Delta(x,0) + \int_0^t \frac{d}{d\tau}V_\Delta(x,\tau)\,d\tau,$$

which, along with (29), yields for all t,

$$V_\Delta(x,0) \geq -\int_0^t \frac{d}{d\tau}V_\Delta(x,\tau)\,d\tau \geq 2\epsilon\int_0^t x(\tau)^T x(\tau)\,d\tau + 2\epsilon\int_0^t p(\tau)^T p(\tau)\,d\tau,$$

from which it follows that $x, p \in \mathbf{L}_2$. As $q = C_q x + D_{qp}p$, we have that $q \in \mathbf{L}_2$.

When it can be proven that x possesses additional regularity properties, it can be concluded from the fact that $x \in \mathbf{L}_2$ that $x(t) \to 0$ as $t \to 0$. This is true, for instance, when Δ is a finite-dimensional uncertainty, or when $D_{qp} = 0$. A more general condition is that the map $\Delta(I - D_{qp}\Delta)^{-1}$ be well-posed. ◇

Remark 13 The condition in the theorem statement that the state of Δ be initialized to zero, roughly speaking, means that there is no initial stored energy in Δ [26, 27, 28]; see Remark 19 to see where this assumption plays a role in the development. $\diamond$

Remark 14 Theorem 3 provides the recipe (27) for constructing such Lyapunov functionals, parametrized by the variables $P = P^T$ and $R = R^T \in \mathcal{M}$ that provide a certificate of stability of system (1) for that Δ. It is important to note that the variables P and R are independent of $\Delta \in \mathbf{\Delta}$, and are only required to satisfy condition (25).

We make the important observation that as (25) is a linear matrix inequality, the set of P and R such that (25) holds is convex. Thus, Theorem 3 provides a *convex parametrization* of a subset of Lyapunov functionals that prove robust stability of system (1) over $\mathbf{\Delta}$. $\diamond$

Remark 15 The Lyapunov functional construction given in Theorem (25) yields the classical Lyapunov-Krasovksii functionals for delay systems [29, 30] as a special case. The LTI system $\frac{d}{dt}x(t) = A_0x(t) + A_1x(t-t_1)$ can be rewritten as

$$\frac{d}{dt}x(t) = Ax(t) + A_0p(t) \quad q(t) = x(t), \quad p(t) = q(t-t_1). \tag{30}$$

Thus $\Delta(s) = e^{-st_1}I$, with $\|\Delta\|_\infty \leq 1$. Clearly $\Delta(j\omega)^*R\Delta(j\omega) - R \leq 0$ for every $n \times n$ matrix positive definite matrix R, so that a subset of $\mathbf{\Pi}$ (see (4) and (5)) is given by

$$\mathbf{\Pi}_0 \stackrel{\Delta}{=} \left\{\Pi : j\mathbf{R} \to \mathbf{R}^{n\times n} : \Pi(j\omega) = R, \text{ with } R = R^T > 0\right\}. \tag{31}$$

It is readily verified that from Theorem 2, a sufficient condition for robust stability of system (30) is given by the LMI

$$P > 0, \;\; R > 0, \;\; \begin{bmatrix} A^TP + PA & PA_0 \\ A_0^TP & 0 \end{bmatrix} + \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix} < 0.$$

The corresponding Lyapunov functional, constructed using Theorem 3, is the Lyapunov-Krasovskii functional

$$V(x,t) = \begin{cases} x(t)^TPx(t) + \int_0^t x(\tau)^TRx(\tau)\,d\tau, & t \in [0, t_1], \\ x(t)^TPx(t) + \int_{t-t_1}^t x(\tau)^TRx(\tau)\,d\tau, & t > t_1. \end{cases}$$

The approach presented in this paper offers the potential for deriving less conservative stability conditions for system (30), when richer subsets of $\mathbf{\Pi}$ (other than $\mathbf{\Pi}_0$ in (31)) are employed. We shall not pursue this point further here. $\diamond$

5 Performance analysis using Lyapunov functionals

It is well-known that Lyapunov functions can be used for robust performance analysis; see for example, the monograph [17]. We now demonstrate how the convex parametrization of Lyapunov functionals, given in Theorem 3, can be used for the analysis of robust performance for LTI systems with structured LTI uncertainties. While a long list of performance measures can be studied using Lyapunov functions or functionals [17], we consider four here, for purposes of illustration. Throughout this section, we assume the state of Δ is initialized to zero.

5.1 Invariant ellipsoids

Given only that $x(0)^T x(0) \leq 1$, suppose that we are interested in a set $\mathcal{E}$ such that irrespective of $x(0)$ and $\Delta \in \mathbf{\Delta}$, the state trajectory of system (1) satisfies $x(t) \in \mathcal{E}$ for all $t \geq 0$.

We can use the parametrization of Lyapunov functionals given by Theorem 3 to define invariant sets $\mathcal{E}$ that are ellipsoids. Suppose the functional $V_\Delta(\cdot,\cdot)$ of the form (27), for every $\Delta \in \mathbf{\Delta}$, satisfies $dV_\Delta(x,t)/dt \leq 0$ for all trajectories of system (1). Partitioning P as

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}, \tag{32}$$

where $P_{11} \in \mathbf{R}^{n\times n}$, we have, for all $t \geq 0$,

$$V_\Delta(x,t) \leq V_\Delta(x,0) = x(0)^T P_{11} x(0).$$

The condition $dV_\Delta(x,t)/dt \leq 0$ for every $\Delta \in \mathbf{\Delta}$ and for all trajectories of system (1) is guaranteed by (25).

Consequently, for all $t \geq 0$,

$$\begin{bmatrix} x(t) \\ x_1(t) \\ x_2(t) \end{bmatrix}^T P \begin{bmatrix} x(t) \\ x_1(t) \\ x_2(t) \end{bmatrix} \leq x(0)^T P_{11} x(0),$$

or

$$x(t)^T \left(P_{11} - P_{12}P_{22}^{-1}P_{12}^T\right) x(t) \leq x(0)^T P_{11} x(0)$$

for all trajectories of system (1) for every $\Delta \in \mathbf{\Delta}$. Suppose that $P_{11} \leq I$, so that $x(0)^T P_{11} x(0) \leq 1$. (This simply represents a normalization of P.) Then, the ellipsoid

$$\mathcal{E} = \left\{\xi \,:\, \xi^T \left(P_{11} - P_{12}P_{22}^{-1}P_{12}^T\right) \xi \leq 1\right\} \tag{33}$$

is an invariant set for the trajectories of system (1) starting in initial conditions that satisfy $x(0)^T x(0) \leq 1$.

This observation offers a number of possibilities. For instance, we can find the minimum volume invariant ellipsoid of the form (33) by minimizing $\log\det\left(P_{11} - P_{12}P_{22}^{-1}P_{12}^T\right)^{-1}$ subject to $P_{11} \le I$ and (25). This is equivalent to the convex optimization problem in variables $Q \in \mathbf{R}^{n\times n}$, $P \in \mathbf{R}^{(n+2n_v)\times(n+2n_v)}$ and $R \in \mathcal{M}$:

$$\begin{array}{ll} \text{minimize:} & \log\det Q^{-1} \\ \text{subject to:} & (25),\ Q > 0,\ P_{11} \le I,\ \left[\begin{array}{cc} P_{11} - Q & P_{12} \\ P_{12}^T & P_{22} \end{array}\right] > 0. \end{array} \tag{34}$$

Other possibilities are maximizing the minimum eigenvalue of $Q = P_{11} - P_{12}P_{22}^{-1}P_{12}^T$ (this is equivalent to minimizing the maximum diameter of $\mathcal{E}$), or $\mathbf{Tr}\ Q$ (which can be interpreted as being inversely proportional to a measure of the size of $\mathcal{E}$).

Remark 16 It is straightforward to adapt the foregoing analysis when invariant ellipsoids are sought not for the entire state vector but only for a few of its components. For example, suppose that we are interested in an invariant ellipsoid for only the first two components of the state (with the rest of the components initialized to zero). The only essential changes required in the above analysis are: $x(0)$ is replaced by a vector comprising the initial conditions of the first two state components, and the partitioning of P in (32) is such that the matrix P_{11} is now a 2×2 matrix. $\diamond$

5.2 Reachable sets

Consider system (1), with zero initial condition, augmented with an input w:

$$\frac{d}{dt}x(t) = Ax(t) + B_p p(t) + B_w w(t),\ x(0) = 0, \quad q(t) = C_q x(t) + D_{qp} p(t), \quad p(t) = \Delta(q,t), \tag{35}$$

where $B_w \in \mathbf{R}^{n\times n_w}$, and $\Delta \in \mathbf{\Delta}$. Suppose we are interested in the set of reachable states with unit-energy inputs for this set of systems, i.e.,

$$\mathcal{R}_{\mathrm{ue}} \triangleq \left\{ x(T)\ :\ \Delta \in \mathbf{\Delta},\ x,\ w \text{ satisfy (35), and } \int_0^T w(t)^T w(t)\, dt \le 1,\ T \ge 0 \right\}.$$

We now show we can bound $\mathcal{R}_{\mathrm{ue}}$ using Lyapunov functionals of the form (27).

Suppose the functional $V_\Delta(\cdot,\cdot)$ of the form (27), for every $\Delta \in \mathbf{\Delta}$, satisfies

$$dV_\Delta(x,t)/dt \le w(t)^T w(t) \tag{36}$$

for all trajectories of system (35). Then, it is easy to show that for all $T \ge 0$, $\int_0^T w(t)^T w(t)\, dt \le 1$ implies that $V_\Delta(x,t) \le 1$. Therefore, for all $T \ge 0$, we have

$$\left[\begin{array}{c} x(T) \\ x_1(T) \\ x_2(T) \end{array}\right]^T P \left[\begin{array}{c} x(T) \\ x_1(T) \\ x_2(T) \end{array}\right] \le 1.$$

Partitioning P as in (32),

$$x(T)^T \left(P_{11} - P_{12} P_{22}^{-1} P_{12}^T \right) x(T) \leq 1,$$

or the ellipsoid

$$\mathcal{E} = \left\{ \xi \; : \; \xi^T \left(P_{11} - P_{12} P_{22}^{-1} P_{12}^T \right) \xi \leq 1 \right\}, \tag{37}$$

contains the reachable set $\mathcal{R}_{\mathrm{ue}}$.

With $\tilde{B}_w^T \triangleq \begin{bmatrix} B_w^T & 0 & 0 \end{bmatrix}^T$ where $\tilde{B}_w \in \mathbf{R}^{(n+2n_v)\times n_w}$, it is readily verified that condition (36) holds for all $\Delta \in \mathbf{\Delta}$ and for all x satisfying (35) if the LMI in the variables $P = P^T$ and $R \in \mathcal{M}$ is feasible:

$$P > 0, \; R > 0, \; \begin{bmatrix} \tilde{A}^T P + P\tilde{A} & P\tilde{B} & P\tilde{B}_w \\ \tilde{B}^T P & 0 & 0 \\ \tilde{B}_w^T P & 0 & -I \end{bmatrix} + \begin{bmatrix} \tilde{M} & 0 \\ 0 & 0 \end{bmatrix} < 0. \tag{38}$$

Thus (37) and (38) provide a parametrization of ellipsoids that contain the reachable set $\mathcal{R}_{\mathrm{ue}}$.

We can find the minimum volume ellipsoid that contains $\mathcal{R}_{\mathrm{ue}}$ by minimizing $\log\det Q^{-1}$ where $Q = P_{11} - P_{12}P_{22}^{-1}P_{12}^T$. This is the convex optimization problem in the variables $Q \in \mathbf{R}^{n\times n}$, $P \in \mathbf{R}^{(n+2n_v)\times(n+2n_v)}$ and $R \in \mathcal{M}$:

$$\begin{array}{ll} \text{minimize:} & \log\det Q^{-1} \\ \text{subject to:} & (38), \;\; Q > 0, \;\; \begin{bmatrix} P_{11} - Q & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix} > 0. \end{array} \tag{39}$$

As with §5.1, other possibilities are maximizing either the minimum eigenvalue of Q, or $\mathbf{Tr}\; Q$.

5.3 Bound on output energy

Consider system (1) augmented with an output z:

$$\frac{d}{dt}x(t) = Ax(t) + B_p p(t), \quad q(t) = C_q x(t) + D_{qp} p(t), \quad p(t) = \Delta(q,t), \quad z(t) = C_z x(t), \tag{40}$$

where $C_z \in \mathbf{R}^{n_z \times n}$, and $\Delta \in \mathbf{\Delta}$. Suppose that we are interested in the maximum output energy given a certain initial state for this set of systems:

$$E_{\max}(x(0)) = \sup_{\Delta \in \mathbf{\Delta}} \left\{ \int_0^\infty z(t)^T z(t)\, dt \right\}. \tag{41}$$

Suppose that there exists $V_\Delta(\cdot,\cdot)$ of the form (27) such that for every $\Delta \in \mathbf{\Delta}$,

$$\frac{d}{dt} V_\Delta(x,t) \leq -z(t)^T z(t) \tag{42}$$

for all trajectories of system (40). Then, partitioning P as in (32), it is easy to show that $E_{\max}(x(0)) \leq x(0)^T P_{11} x(0)$. With $\tilde{C}_z \triangleq \begin{bmatrix} C_z & 0 & 0 \end{bmatrix}$, where $\tilde{C}_z \in \mathbf{R}^{n_z\times(n+2n_v)}$, it is readily verified that condition (42) holds for all $\Delta \in \mathbf{\Delta}$ and for all x satisfying (40) if the LMI in the variables $P = P^T$ and $R \in \mathcal{M}$ is feasible:

$$P > 0,\ R > 0,\ \begin{bmatrix} \tilde{A}^T P + P\tilde{A} + \tilde{C}_z^T \tilde{C}_z & P\tilde{B} \\ \tilde{B}^T P & 0 \end{bmatrix} + \tilde{M} < 0. \tag{43}$$

The best upper bound on the output energy can be obtained by solving the convex optimization problem in variables $P \in \mathbf{R}^{(n+2n_v)\times(n+2n_v)}$ and $R \in \mathcal{M}$:

$$\begin{array}{ll} \text{minimize:} & x(0)^T P_{11} x(0) \\ \text{subject to:} & (43). \end{array} \tag{44}$$

5.4 Bound on the $\mathbf{H}_2$ norm

Consider system (1), with zero initial condition, augmented with an input w and an output z:

$$\frac{d}{dt}x(t) = Ax(t) + B_p p(t) + B_w w(t), \quad q(t) = C_q x(t) + D_{qp} p(t), \quad z(t) = C_z x(t), \quad p(t) = \Delta(q,t), \tag{45}$$

where $B_w \in \mathbf{R}^{n\times n_w}$, $C_z \in \mathbf{R}^{n_z\times n}$, and $\Delta \in \mathbf{\Delta}$.

Suppose that we are interested in the largest $\mathbf{H}_2$ norm of the transfer function from w to z,

$$\|H\|_{2,\mathrm{wc}} = \sup_{\Delta \in \mathbf{\Delta}} \|H_{zw}\|_2. \tag{46}$$

Recall that for a strictly proper LTI system with transfer function $H(s)$, the $\mathbf{H}_2$ norm satisfies

$$\|H\|_2^2 = \int_0^\infty \mathrm{Tr}\ h(t)^T h(t)\, dt.$$

With (A, B, C) being a state-space realization of H, we have

$$\|H\|_2^2 = \sum_{i=1}^{n_w} \int_0^\infty (Ce^{At}b_i)^T (Ce^{At}b_i)\, dt,$$

where $b_1, \ldots, b_{n_w}$ denote the columns of B_w. Thus, the $\mathbf{H}_2$ norm can be re-interpreted as the square-root of the sum of the energy in the output z for the system $\dot{x} = Ax$, $z = Cx$, over the initial conditions $x(0) = b_i$.

Returning to system (45), we thus have an immediate upper bound on the quantity $\|H\|_{2,\mathrm{wc}}$ in (46): We simply combine the bound on the output energy derived in §5.3, with the interpretation of the $\mathbf{H}_2$ norm as the square-root of the sum of output energy for initial conditions b_i. Therefore,

$$\|H\|_{2,\mathrm{wc}}^2 \le \sum_{i=1}^{n_w} \sup_{\Delta \in \mathbf{\Delta}} \left\{ \int_0^\infty z_i(t)^T z_i(t)\, dt \right\},$$

where

$$\frac{d}{dt}x(t) = Ax(t) + B_p p(t), \quad x(0) = b_i,$$
$$q(t) = C_q x(t) + D_{qp} p(t), \quad z_i(t) = C_z x(t), \quad p(t) = \Delta(q,t). \tag{47}$$

Following the development in §5.3, an upper bound on $\|H\|_{2,\mathrm{wc}}^2$ can be obtained by solving the convex optimization problem in the variables $R^{(i)} \in \mathcal{M}$, γ_i and $P^{(i)} = (P^{(i)})^T$, $i = 1,\ldots,n_w$:

$$\begin{array}{ll} \text{minimize:} & \sum_{i=1}^{n_w} \gamma_i \\ \text{subject to:} & \text{for } i = 1,\ldots,n_w, \end{array}$$

$$\begin{bmatrix} \tilde{A}^T P^{(i)} + P^{(i)}\tilde{A} + \tilde{C}_z^T \tilde{C}_z & P^{(i)}\tilde{B} \\ \tilde{B}^T P^{(i)} & 0 \end{bmatrix} + \tilde{M} < 0, \tag{48}$$
$$P^{(i)} > 0, \quad R^{(i)} > 0, \quad \gamma_i \geq b_i^T P_{11}^{(i)} b_i.$$

($P_{11}^{(i)}$ is the principal $n \times n$ block of $P^{(i)}$, and $\tilde{C}_z \triangleq \begin{bmatrix} C_z & 0 & 0 \end{bmatrix}$, where $\tilde{C}_z \in \mathbf{R}^{n_z \times (n+2n_v)}$.)

Remark 17 The bound on the $\mathbf{H}_2$ norm derived here is closest in spirit to the one derived in [31], applicable when the uncertainties Δ are diagonal; there, rather than deriving Lyapunov functionals, the author considers the multipliers Π directly, and interprets its unstable modes as corresponding to an anti-causal system. Bounds on $\mathbf{H}_2$ norm are then derived using optimal control theory. For an overview to the problem of the "robust $\mathbf{H}_2$ analysis", see [32]. $\diamond$

5.5 A numerical example

We present a simple numerical example that illustrates the development in §5.1–5.4. Consider a specific instance of system (45) with

$$A = \begin{bmatrix} -0.30 & 0 & -0.10 & 0 & -0.10 \\ 0.10 & -0.10 & 0 & 0 & 0.20 \\ 0 & 0 & -0.30 & 0 & -0.40 \\ -0.40 & 0.40 & -0.20 & -1.00 & -0.30 \\ 0 & 0 & 0.30 & -0.30 & -0.20 \end{bmatrix} \quad B_p = \begin{bmatrix} 0.04 & 0.06 & -0.05 \\ 0.60 & 0.07 & 0.01 \\ 0.08 & -0.46 & -0.06 \\ 0.02 & -0.11 & 0.13 \\ -0.12 & -0.07 & -0.14 \end{bmatrix};$$
$$C_q = \begin{bmatrix} 0.08 & 0.05 & 0.02 & 0.08 & 0.06 \\ 0.08 & -0.01 & 0.11 & -0.09 & -0.06 \\ 0.08 & 0.05 & 0.07 & 0.04 & -0.06 \end{bmatrix}; \quad D_{qp} = \begin{bmatrix} -0.053 & 0 & 1.080 \\ 0.590 & -0.053 & -0.290 \\ 0.290 & -0.100 & 0.077 \end{bmatrix};$$
$$C_z = \begin{bmatrix} 0 & 0 & 0 & 2 & 0 \end{bmatrix}; \quad B_w = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

The uncertainty set $\mathbf{\Delta}$ is defined by

$$\mathbf{\Delta} \triangleq \left\{ \Delta \ : \ \Delta(s) = \begin{bmatrix} \delta^c(s) I & \\ & \Delta^C(s) \end{bmatrix}, \ \delta^c : \mathbf{C} \to \mathbf{C}, \ \Delta^C : \mathbf{C} \to \mathbf{C}, \ \|\Delta\|_\infty \leq 1 \right\}, \tag{49}$$

where the identity matrix is of size 2. (That is, Δ is block-diagonal, with a scaled identity block of size two, and a scalar uncertainty). We use a parametrization for $\mathbf{\Pi}_{\mathrm{fin}}$ given by (17–20).

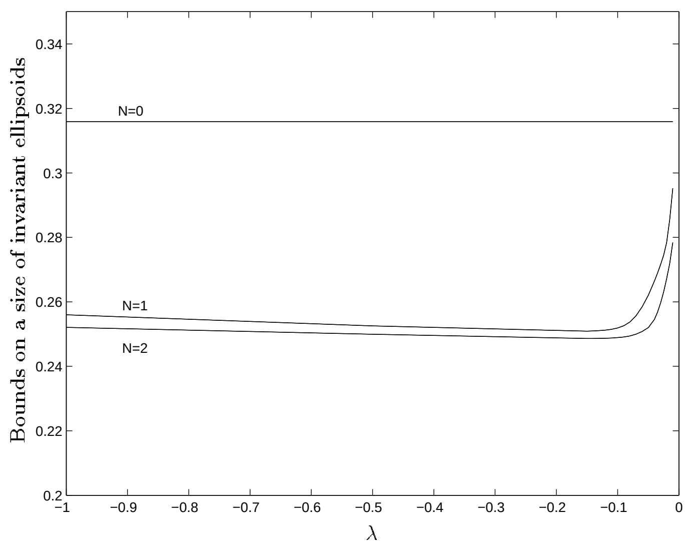
For this example, we present numerical results obtained via the solution of the optimization problems discussed in §5.1–5.4. All the numerical optimization problems were solved in MATLAB using the LMI control toolbox [19]. In each case of robust performance analysis, we also performed Monte Carlo simulations and picked out a "worst-case" uncertainty; when presenting robust performance upper bounds, we also present the performance of the closed-loop system with the empirically-determined worst-case Δ, which yields a *lower bound* on robust performance measures.

- **Invariant ellipsoids.** We consider the problem of bounding the states of system (1) over all $\Delta \in \mathbf{\Delta}$ and over all initial conditions $x(0)$ satisfying $\|x(0)\|_2 \leq 1$. Figure 2(a) shows the inverse of $\mathbf{Tr}\ (Q)$ corresponding to the optimal invariant ellipsoid (see §5.1 for notation) as a function of λ, for $N = 0, 1$, and 2. (Recall that λ and N are the two variables that parametrize $\mathbf{\Pi}_{\mathrm{fin}}$; see (17–20)).

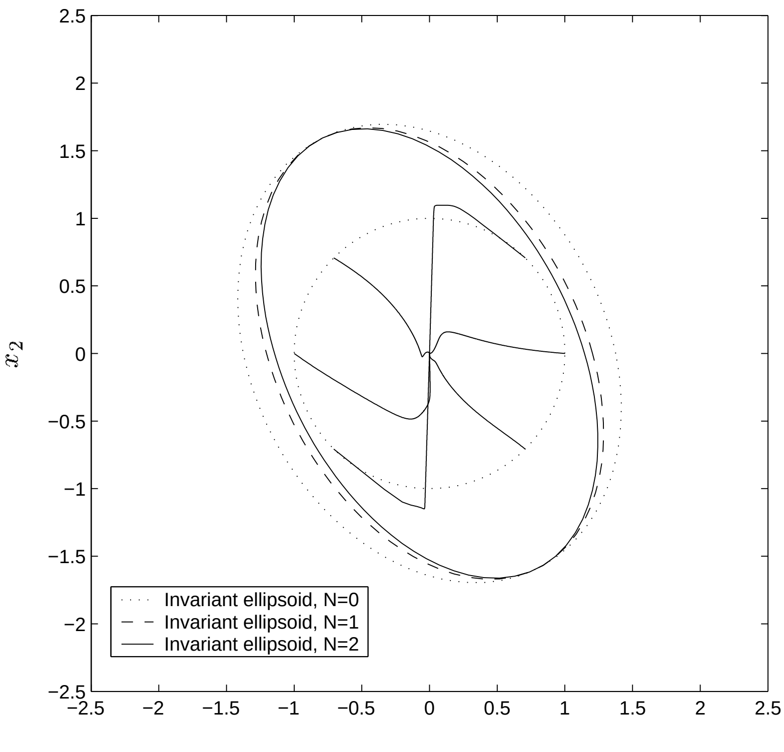
  In order to visualize the invariant ellipsoids, we also consider another problem where only the first two components $\hat{x}(t) = \begin{bmatrix} x_1(t) & x_2(t) \end{bmatrix}^T$ of the state are of interest. Figure 2(b) depicts three optimal invariant ellipsoids for $\hat{x}(t)$, obtained with $N = 0, 1$, and 2 respectively, with $\lambda = -0.1$. Also shown are state trajectories for a few initial conditions, for a few instances of the uncertainty $\Delta(s)$.

- **Worst-case reachable sets.** We consider ellipsoidal bounds for $\mathcal{R}_{\mathrm{ue}}$, the set of states reachable with unit energy inputs for system (35). Figure 3 shows the optimal maximum diameters of the ellipsoids containing $\mathcal{R}_{\mathrm{ue}}$ as a function of λ, for $N = 0, 1$, and 2. The lower bound on the maximum diameters of the reachable ellipsoid, determined using the worst-case Δ determined via Monte Carlo simulations, is 11.68. The best upper bound determined using Lyapunov functionals, is 11.69 (see Figure 3). The relative gap between the upper and lower bounds is about 0.01(0.1%).

- **Worst-case output energy.** Next, we consider bounds on the worst-case output energy for system (40), for the initial condition $x(0) = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}^T$. Figure 4 shows the optimal bound as a function of λ, for $N = 0, 1$, and 2. The lower bound on the worst-case output energy, determined using the worst-case Δ determined via Monte Carlo simulations, is 127. The best upper bound determined using Lyapunov functionals, is 230 (see Figure 4). The gap between the upper and lower bounds is at most 113 (89%).



(a) Bound on a size of invariant ellipsoids versus λ, for various values of N.



(b) Optimal invariant ellipsoids for $\hat{x}(t)$ obtained with $N = 0$ (shown in a dotted line), $N = 1$ (shown in a dashed line), and $N = 2$ (shown in a solid line), with $\lambda = -0.1$. The initial conditions are restricted to lie inside the unit circle (shown in a dotted line). Also shown are state trajectories for a few initial conditions, for a few instances of the uncertainty $\Delta(s)$.

Figure 2: Invariant ellipsoids.

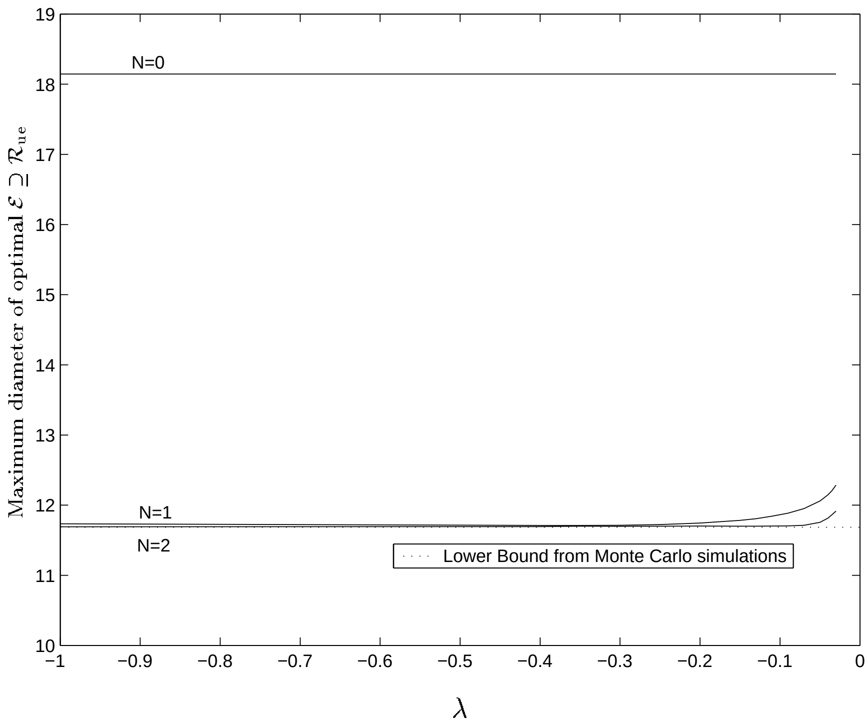


Figure 3: The optimal maximum diameters of ellipsoids containing $\mathcal{R}_{\mathbf{ue}}$ as a function of λ, for various values of N.

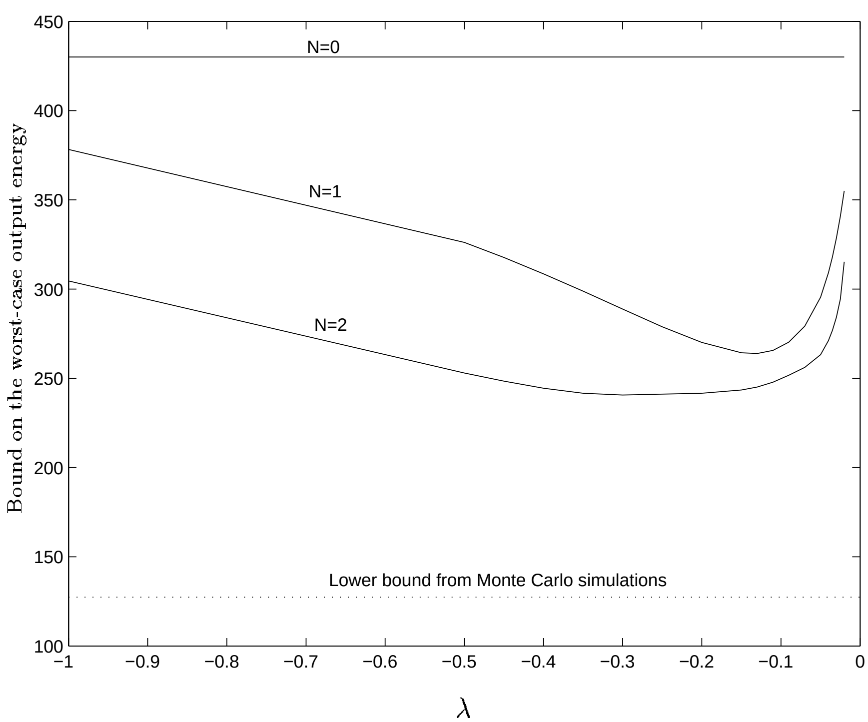


Figure 4: The optimal upper bound on the worst-case output energy, as a function of λ, for various values of N.

- **Worst-case $\mathbf{H}_2$ norm (Robust $\mathbf{H}_2$ performance).** Figure 5 shows the optimal bounds on the worst-case $\mathbf{H}_2$ norm from w to z for system (45), as a function of λ, for $N = 0, 1$, and 2. The lower bound on the worst-case $\mathbf{H}_2$ norm, determined using the worst-case Δ determined via Monte Carlo simulations, is 8.26. The best upper bound determined using Lyapunov functionals, is 8.58 (see Figure 5). The gap between the upper and lower bounds is at most 0.32(4%).

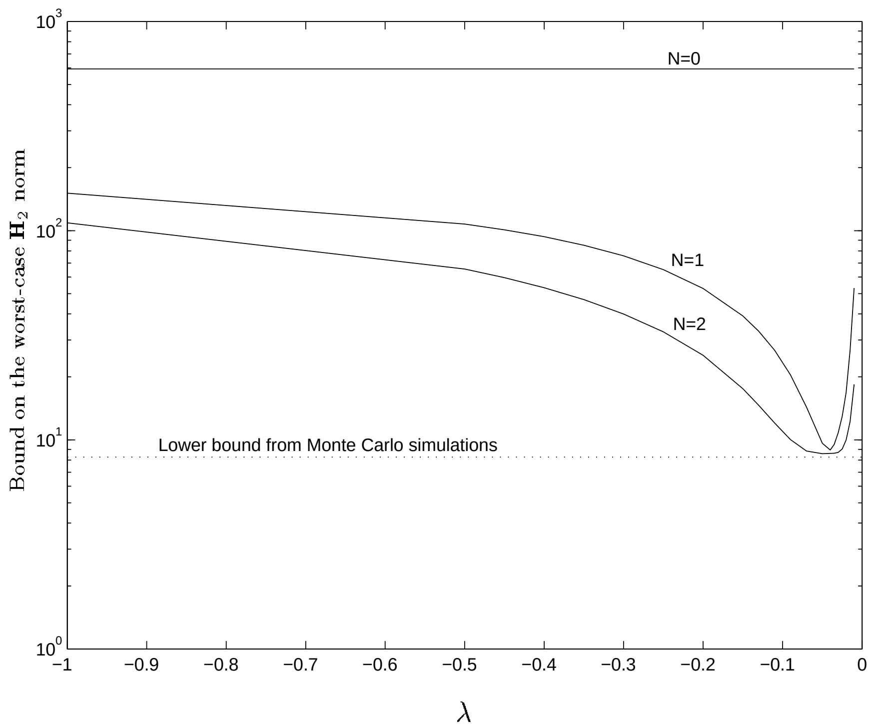


Figure 5: The optimal upper bound on the worst-case $\mathbf{H}_2$ norm from w to z, as a function of λ, for various values of N.

For purposes of comparison, we consider another example, taken from [31], consisting of a five-state system with a scalar LTI uncertainty. (As mentioned in Remark 17, the approach in in [31] is also based on stability multipliers.) The results from [31] and the bound from our algorithm, all with $\lambda = -1$, are shown below:

	$N = 0$	$N = 1$	$N = 2$
Bound from [31]	341.17	137.87	99.03
Our bound	305.05	123.31	88.58

5.6 Discussion

We observe the following from the numerical example presented in section §5.5:

- Significant improvement can be achieved in robust performance analysis with the use of the Lyapunov functionals presented in §4 (the plots corresponding to $N = 1$ and $N = 2$ in Figures 2–5), as compared to traditional analysis using quadratic Lyapunov functions (the plots corresponding to $N = 0$ in Figures 2–5). An order-of-magnitude improvement can be seen in some cases.

- There are two parameters λ and N that influence the form of the Lyapunov functional, and hence the robust performance bounds. Recall that λ determines the expansion point for the Laurent series of the multiplier Π, and that N is the number of terms that are retained in the series expansion. For the example presented in §5.5, it appears that the sensitivity of the bounds to λ varies with the particular application; for the problems of finding invariant ellipsoids and bounding $\mathcal{R}_{\mathrm{ue}}$, the bounds are fairly insensitive to λ, provided it is not too small in magnitude. For the problems of bounding the worst-case output energy and the worst-case $\mathbf{H}_2$ norm, the bounds depend on λ. Note however that for every λ, we do obtain a valid performance bound; the simple technique of gridding over λ can be used to obtain the tightest bound.

  As expected, the bounds become better with increasing N; however, for this example, it is evident that increasing N beyond 2 will not yield a significant improvement in the bound; for a theoretical study of this issue in a closely-related context, see [21].

- The question of the "gap" between the actual robust performance measure and the upper bounds derived herein remains. In §5.5, we have used Monte Carlo simulations to numerically study this gap. For the problems of bounding reachable sets and bounding the worst-case $\mathbf{H}_2$ norm, this gap is virtually nonexistent, indicating that our robust performance bounds are excellent. However, for the problems of bounding the worst-case output energy, this gap is significant, although it is much smaller than the gap with the bound derived using the traditional quadratic Lyapunov functions.

  The theoretical study of the size of the gap is similar in spirit to the simpler study of the gap between μ and its upper bound; unfortunately, S. Treil has proven that the answer to latter question is that the gap in general may be infinite [33] (although it is zero for some special cases [34, §6], and is usually small in practice). We conjecture similar answers to the robust performance question as well.

6 Conclusions

We have derived a convex parametrization of a subset of Lyapunov functionals that prove robust stability of LTI systems with structured LTI uncertainties. The technical difficulty in deriving such a parametrization stems from the fact that the Kalman-Yakubovich-Popov Lemma does not necessarily provide positive-definite certificates. We have tackled this issue by showing that given a stability multiplier, one can always obtain a realization for it wherein the KYP Lemma does provide positive-definite certificates. This result is of independent interest, and proves crucial in our Lyapunov functional construction. We have also shown how the Lyapunov functionals can be immediately used in performance analysis. We believe that these results can be of considerable practical interest.

Several questions remain open, however. First, reformulating frequency-domain inequalities that are encountered in control into convex optimization problems based on LMIs using the KYP Lemma, while theoretically pleasing, leads to numerical optimization problems of quickly growing size. Some efforts are being devoted currently to tackling this issue in other settings [35]. A second issue is that of the choice of multiplier dynamics (see equations (17–20)); the quality of the results obtained using multiplier theory does depend on the choice of multiplier dynamics, and a theoretical and/or empirical study of this dependence will be valuable. A third question is that of the gap between the upper bounds for robust performance measures derived using Lyapunov functionals and the actual values of the performance measures. A final, larger issue is that of robust synthesis: Several existing robust stability analysis techniques have been extended to stabilizing controller synthesis; see for example, [5, 7, 23]; it remains to extend the Lyapunov-functional-based analysis results presented in this paper to the problem of control synthesis with robust performance constraints.

Acknowledgments

The author would like to thank André Tits, whose suggestions helped to improve the paper considerably. The author would also like to thank Eric Feron, Fan Wang and Anders Rantzer for valuable discussions.

References

[1] K. Zhou with J. Doyle and K. Glover, *Robust and Optimal Control*, Prentice Hall, 1996.

[2] M. Green and D. J. N. Limebeer, *Linear Robust Control*, Information and System sciences. Prentice Hall, Englewood Cliffs, NJ, 1995.

[3] J. Doyle, "Analysis of feedback systems with structured uncertainties," *IEE Proc.*, vol. 129-D, no. 6, pp. 242–250, Nov. 1982.

[4] M. G. Safonov, "Stability margins of diagonally perturbed multivariable feedback systems," *IEE Proc.*, vol. 129-D, pp. 251–256, 1982.

[5] G. Balas, J. C. Doyle, K. Glover, A. Packard, and Roy Smith, *μ-analysis and synthesis*, MUSYN, inc., and The Mathworks, Inc., 1991.

[6] C. T. Lawrence, A. L. Tits, and P. Van Dooren, "A fast algorithm for the computation of an upper bound on the μ-norm," *Automatica*, vol. 36, no. 3, pp. 449–456, Mar. 2000.

[7] M. G. Safonov and R. Y. Chiang, "Real/complex K_m-synthesis without curve fitting," in *Control and Dynamic Systems*, C. T. Leondes, Ed., vol. 56, pp. 303–324. Academic Press, New York, 1993.

[8] J. P. How and S. R. Hall, "Connections between the Popov stability criterion and bounds for real parametric uncertainty," in *Proc. American Control Conf.*, June 1993, vol. 2, pp. 1084–1089.

[9] V. Balakrishnan, "Linear matrix inequalities in robustness analysis with multipliers," *Syst. Control Letters*, vol. 25, no. 4, pp. 265–272, 1995.

[10] A. Megretski and A. Rantzer, "System analysis via integral quadratic constraints," *IEEE Trans. Aut. Control*, vol. 42, no. 6, pp. 819–830, June 1997.

[11] U. Jönsson, *Robustness Analysis of Uncertain and Nonlinear Systems*, Ph.D. thesis, Lund Institute of Technology, Sweden, 1996.

[12] V. Balakrishnan and F. Wang, "Efficient computation of a guaranteed lower bound on the robust stability margin for a class of uncertain systems," *IEEE Trans. Aut. Control*, vol. AC-44, no. 11, pp. 2185–2190, Nov. 1999.

[13] J. C. Willems, "On the existence of a nonpositive solution to the Riccati equation," *IEEE Trans. Aut. Control*, vol. 19, pp. 592–593, 1974.

[14] A. Rantzer, "On the Kalman-Yacubovich-Popov lemma," *Syst. Control Letters*, vol. 28, no. 1, pp. 7–10, 1996.

[15] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties*, Academic Press, New York, 1975.

[16] V. Balakrishnan and S. Boyd, "Global optimization in control system analysis and design," in *Control and Dynamic Systems: Advances in Theory and Applications*, C.T. Leondes, Ed., vol. 53. Academic Press, New York, New York, 1992.

[17] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in System and Control Theory*, vol. 15 of *Studies in Applied Mathematics*, SIAM, Philadelphia, PA, June 1994.

[18] L. Vandenberghe and V. Balakrishnan, "Algorithms and software for LMI problems in control," *IEEE Control Syst. Mag.*, pp. 89–95, Oct. 1997.

[19] P. Gahinet, A. Nemirovskii, A. Laub, and M. Chilali, *The LMI Control Toolbox*, The Math-Works, Inc., 1995.

[20] B. A. Francis, *A course in* $\mathbf{H}_\infty$ *Control Theory*, vol. 88 of *Lecture Notes in Control and Information Sciences*, Springer-Verlag, 1987.

[21] Y. S. Chou, A. L. Tits, and V. Balakrishnan, "Stability multipliers and μ upper bounds: Connections and implications for numerical verification of frequency domain conditions," *IEEE Trans. Aut. Control*, vol. AC-44, no. 5, pp. 906–913, May 1999.

[22] W. Haddad and D. Bernstein, "Explicit construction of quadratic Lyapunov functions for the small gain, positivity, circle, and Popov theorems and their application to robust stability. part I: Continuous-time theory," *Int. J. Robust and Nonlinear Control*, vol. 3, pp. 313–339, 1993.

[23] J. P. How, S. R. Hall, and W. M. Haddad, "Robust controllers for the middeck active control experiment using Popov controller synthesis," *IEEE Trans. Control Sys. Tech.*, vol. 2, no. 2, pp. 73–87, June 1994.

[24] M. Fu and S. Dasgupta, "Parametric Lyapunov function for uncertain system: the multiplier approach," in *Advances in Linear Matrix Inequality Methods in Control*, L. El Ghaoui and S.-I. Niculescu, Eds., Advances in Control and Design, chapter 5. SIAM, Philadelphia, PA, 2000.

[25] J. P. LaSalle, "Some extensions of Liapunov's second method," *IRE Trans. Circuit Theory*, pp. 520–527, Dec. 1960.

[26] J. C. Willems, "Least squares stationary optimal control and the algebraic Riccati equation," *IEEE Trans. Aut. Control*, vol. AC-16, no. 6, pp. 621–634, Dec. 1971.

[27] J. C. Willems, "Dissipative dynamical systems. Part I: General theory," *Archive for Rational Mechanics and Analysis*, vol. 45, pp. 321–351, 1972.

[28] J. C. Willems, "Dissipative dynamical systems. Part II: Linear systems with quadratic supply rates," *Archive for Rational Mechanics and Analysis*, vol. 45, pp. 352–393, 1972.

[29] N. N. Krasovksii, *Stability of Motion*, Stanford University Press, 1963.

[30] S. I. Niculescu, E. I. Verriest, L. Dugard, and J. M. Dion, "Stability and robust stability of time-delay systems : A guided tour," in *Stability and Control of Time-delay Systems*, L. Dugard L and E. I. Verriest, Eds., number 228 in Lectures Notes in Control and Information Science, chapter 1, pp. 1–71. Springer-Verlag, London, 1998.

[31] E. Feron, "Analysis of robust $\mathbf{H}_2$ performance using multiplier theory," *SIAM J. on Control*, vol. 35, no. 1, pp. 160–177, Jan. 1997.

[32] F. Paganini and E. Feron, "LMI methods for robust $\mathbf{H}_2$ analysis: A survey with comparisons," in *Advances in Linear Matrix Inequality Methods in Control*, L. El Ghaoui and S.-I. Niculescu, Eds., Advances in Control and Design. SIAM, 1999.

[33] S. Treil, "The gap between complex structured singular value μ and its upper bound is infinite," 2001, Preprint, available at `http://www.math.brown.edu/~treil/papers/mu/mu-abs.html`.

[34] A. Packard and J. Doyle, "The complex structured singular value," *Automatica*, vol. 29, no. 1, pp. 71–109, 1993.

[35] B. Alkire and L. Vandenberghe, "Interior-point methods for magnitude filter design," in *Proc. IEEE 26th Intl. Conf. on Acoustics, Speech and Signal Processing (ICASSP)*, Salt Lake City, Utah, May 2001.

A Proof of Lemma 4

We begin with the following Lemma, which was stated without proof by Willems in [13]. We provide a short proof here for completeness.

Lemma 5 *Let H_1 and H_2 be stable transfer functions with minimal state-space realizations (A_1, B_1, C_1, D_1) and (A_2, B_2, C_2, D_2) respectively. Moreover, let H_2^{-1} be stable. Suppose for some $\epsilon > 0$, for all $\omega \in \mathbf{R}$,*

$$H_1(j\omega)^* H_1(j\omega) - H_2(j\omega)^* H_2(j\omega) \le -2\epsilon I.$$

Let

$$\hat{A} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \hat{M} = \begin{bmatrix} C_1^T & 0 \\ 0 & C_2^T \\ D_1^T & D_2^T \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} C_1 & 0 & D_1 \\ 0 & C_2 & D_2 \end{bmatrix}. \tag{50}$$

Then:

1. *There exists $P = P^T$ such that*
$$\begin{bmatrix} \hat{A}^T P + P\hat{A} & P\hat{B} \\ \hat{B}^T P & 0 \end{bmatrix} + \hat{M} < 0, \tag{51}$$

2. *Every P that satisfies (51) is positive-definite.*

Proof: Item (1) follows immediately from the KYP Lemma (Lemma 1). We now prove item (2).

Suppose that $P = P^T$ satisfies (51). Then, applying a congruence transformation to (51), we have

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -D_2^{-1}C_2 & D_2^{-1} \end{bmatrix}^T \left(\begin{bmatrix} \hat{A}^T P + P\hat{A} & P\hat{B} \\ \hat{B}^T P & 0 \end{bmatrix} + \hat{M} \right) \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & -D_2^{-1}C_2 & D_2^{-1} \end{bmatrix} < 0,$$

which can be rewritten as

$$\begin{bmatrix} \bar{A}^T P + P\bar{A} & P\bar{B} \\ \bar{B}^T P & 0 \end{bmatrix} + \begin{bmatrix} \bar{C}^T & 0 \\ \bar{D}^T & I \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \begin{bmatrix} \bar{C} & \bar{D} \\ 0 & I \end{bmatrix} < 0, \tag{52}$$

where

$$\bar{A} = \begin{bmatrix} A_1 & -B_1 D_2^{-1} C_2 \\ 0 & A_2 - B_2 D_2^{-1} C_2 \end{bmatrix}, \quad \bar{B} = \begin{bmatrix} B_1 D_2^{-1} \\ B_2 D_2^{-1} \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C_1 & -D_1 D_2^{-1} C_2 \end{bmatrix}, \quad \bar{D} = D_1 D_2^{-1}.$$

It is readily verified that $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ is a state-space realization of $H_1(s)H_2^{-1}(s)$, and consequently, all eigenvalues of $\bar{A}$ have negative real part. Then, it follows from (52) that $P > 0$.

□

Lemma 6 *Suppose that there exists $\tilde{R} = \tilde{R}^T \in \mathcal{M}$ such that for some $\epsilon > 0$, $\Psi(j\omega)^* \tilde{R} \Psi(j\omega) \ge 2\epsilon I$. Then, there exists $\hat{R} = \hat{R}^T \in \mathcal{M}$, with $\hat{R} \ge 0$ such that*

$$\Psi(j\omega)^* \tilde{R} \Psi(j\omega) = \Psi(j\omega)^* \hat{R} \Psi(j\omega)$$

for all $\omega \in \mathbf{R}$.

Proof: Let $U(j\omega)$ be a spectral factor of $\Psi(j\omega)^*\tilde{R}\Psi(j\omega)$, i.e., $U : \mathbf{C} \to \mathbf{C}^{m\times m}$, analytic in the right-half complex plane, with an inverse that is also analytic in the right-half complex plane, and with

$$\Psi(j\omega)^*\tilde{R}\Psi(j\omega) = U(j\omega)^*U(j\omega) \tag{53}$$

for all $\omega \in \mathbf{R}$. (The existence of a spectral factor is guaranteed by the positivity of $\Psi(j\omega)^*\tilde{R}\Psi(j\omega)$ on the imaginary axis.) Recall that

$$\Psi(j\omega) = \left[\begin{array}{cccc} \Psi_0(j\omega)^* & \Psi_1(j\omega)^* & \cdots & \Psi_N(j\omega)^* \end{array}\right]^*, \text{ with } \Psi_i(j\omega) = \frac{1}{(j\omega-\lambda)^i}I,$$

so that we must have

$$U(j\omega) = \sum_{i=0}^{N} \frac{1}{(j\omega-\lambda)^i}G_i,$$

with $G_i \in \mathcal{M}$, $i = 0,1,\ldots,N$. Thus, $U(s) = G\Psi(s)$, where $G \in \mathbf{R}^{m\times m(N+1)}$ is given by

$$G = \left[\begin{array}{ccc} G_0 & G_1 & \cdots G_N \end{array}\right], \tag{54}$$

Returning to (53), we have

$$\Psi(j\omega)^*\tilde{R}\Psi(j\omega) = \Psi(j\omega)^*G^TG\Psi(j\omega),$$

where $G^TG \in \mathcal{M}$. Setting $\hat{R} = G^TG$ completes the proof.

□

Remark 18 Lemma 6 can be re-interpreted in view of Remark 8. Partition $\tilde{R}$ in Lemma 6 as

$$\tilde{R} = \left[\begin{array}{cc} \tilde{R}_{11} & \tilde{R}_{12} \\ \tilde{R}_{12}^T & \tilde{R}_{22} \end{array}\right],$$

where $\tilde{R}_{22} \in \mathbf{R}^{m\times m}$. It is easily argued that $D_\psi^T\tilde{R}_{22}D_\psi$ is invertible. Let X_{m} be the stabilizing solution solution (see Remark 8) to the algebraic Riccati equation

$$A_\psi^TX_{\mathrm{m}} + X_{\mathrm{m}}A_\psi - C_\psi^T\tilde{R}_{11}C_\psi + (X_{\mathrm{m}}B_\psi - C_\psi^T\tilde{R}_{12}D_\psi)(D_\psi^T\tilde{R}_{22}D_\psi)^{-1}(X_{\mathrm{m}}B_\psi - C_\psi^T\tilde{R}_{12}D_\psi)^T = 0.$$

Then, with $C_{\mathrm{sf}} = -(D_\psi^TM_{22}D_\psi)^{-1/2}(B_\psi^TX_{\mathrm{m}} - D_\psi^TM_{12}^TC_\psi)$ and $D_{\mathrm{sf}} = (D_\psi^TM_{22}D_\psi)^{1/2}$, we have from Remark 8 that

$$\Psi(j\omega)^*\tilde{R}\Psi(j\omega) = (C_{\mathrm{sf}}(j\omega I - A_\psi)^{-1}B_\psi + D_{\mathrm{sf}})^*(C_{\mathrm{sf}}(j\omega I - A_\psi)^{-1}B_{\mathrm{sf}} + D_{\mathrm{sf}}).$$

The claim proved in Lemma 6 is that we can find G of the form (54), with

$$G\left[\begin{array}{cc} C_\psi & D_\psi \end{array}\right] = \left[\begin{array}{cc} C_{\mathrm{sf}} & D_{\mathrm{sf}} \end{array}\right].$$

◇

We are now ready to prove Lemma 4. We prove that (1) $\iff$ (2), (1) $\Rightarrow$ (3), and (3) $\Rightarrow$ (2).

To show (2) $\Rightarrow$ (1) is easy. Since $R = R^T > 0$, it immediately follows that $\Pi \in \mathbf{\Pi}_{\text{fin}}$. That (3) $\Rightarrow$ (2) follows directly from Lemma 1.

Consider (1) $\Rightarrow$ (2), and (1) $\Rightarrow$ (3). Suppose that there exists $\tilde{R} = \tilde{R}^T \in \mathcal{M}$ such that with $\Pi(j\omega) = \Psi(j\omega)^*\tilde{R}\Psi(j\omega)$, for some $\epsilon > 0$, $\Pi(j\omega) \geq 2\epsilon I$, and such that for some $\epsilon > 0$, inequality (7) holds. Then, from Lemma 6, there exists $\hat{R} = \hat{R}^T \in \mathcal{M}$, with $\hat{R} \geq 0$ such that $\Pi(j\omega) = \Psi(j\omega)^*\hat{R}\Psi(j\omega)$ for all $\omega \in \mathbf{R}$. Define $R = \hat{R} + \mu I$ where $\mu > 0$. Then, $0 < R^T = R \in \mathcal{M}$, and we have

$$\begin{array}{l}
H(j\omega)^*\Psi(j\omega)^*R\Psi(j\omega)H(j\omega) - \Psi(j\omega)^*R\Psi(j\omega) \\
= H(j\omega)^*\Psi(j\omega)^*\tilde{R}\Psi(j\omega)H(j\omega) - \Psi(j\omega)^*\tilde{R}\Psi(j\omega) \\
\qquad\qquad\qquad\qquad +\mu\left(H(j\omega)^*\Psi(j\omega)^*\Psi(j\omega)H(j\omega) - \Psi(j\omega)^*\Psi(j\omega)\right).
\end{array}$$

By assumption, for some $\epsilon > 0$, we have that $H(j\omega)^*\Psi(j\omega)^*\tilde{R}\Psi(j\omega)H(j\omega) - \Psi(j\omega)^*\tilde{R}\Psi(j\omega) \leq -2\epsilon I$ for all $\omega \in \mathbf{R}$. Therefore, for some $\mu > 0$ that is small enough, there exists some $\epsilon > 0$ such that

$$H(j\omega)^*\Psi(j\omega)^*R\Psi(j\omega)H(j\omega) - \Psi(j\omega)^*R\Psi(j\omega) \leq 2\epsilon I,$$

for all $\omega \in \mathbf{R}$. This proves that (1) $\Rightarrow$ (2).

Proceeding further, from Lemma 1, there exists $P = P^T$ such that

$$\begin{bmatrix} \tilde{A}^T P + P\tilde{A} & P\tilde{B} \\ \tilde{B}^T P & 0 \end{bmatrix} + \begin{bmatrix} \tilde{C}^T \\ \tilde{D}^T \end{bmatrix} \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix} \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix} < 0$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{C}$ are defined in (21). Equivalently,

$$\begin{bmatrix} \tilde{A}^T P + P\tilde{A} & P\tilde{B} \\ \tilde{B}^T P & 0 \end{bmatrix} + \begin{bmatrix} \tilde{C}^T \\ \tilde{D}^T \end{bmatrix} \left(\begin{bmatrix} G^T G & 0 \\ 0 & -G^T G \end{bmatrix} + \mu \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix} \right) \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix} < 0.$$

From continuity arguments, for μ small enough, if follows that

$$\begin{bmatrix} \tilde{A}^T P + P\tilde{A} & P\tilde{B} \\ \tilde{B}^T P & 0 \end{bmatrix} + \begin{bmatrix} \tilde{C}^T \\ \tilde{D}^T \end{bmatrix} \begin{bmatrix} G^T G & 0 \\ 0 & -G^T G \end{bmatrix} \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix} < 0. \tag{55}$$

Define $H_1(s) = G\Psi(s)H(s)$ and $H_2(s) = G\Psi(s)$, Recall that $G\Psi(s)$ is a spectral factor of $\Pi(j\omega)$, so that H_1, H_2 and H_2^{-1} are stable. Moreover, with

$$A_1 = \begin{bmatrix} A & 0 \\ B_\psi C_q & A_\psi \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_p \\ B_\psi D_{qp} \end{bmatrix}, \quad C_1 = G \begin{bmatrix} D_\psi C_q & C_\psi \end{bmatrix}, \quad D_1 = G D_\psi D_{qp},$$

$A_2 = A_\psi$, $B_2 = B_\psi$, $C_2 = GC_\psi$, $D_2 = GD_\psi$, and $\hat{A}$, $\hat{B}$, and $\hat{M}$ as defined in (50), inequality (55) is equivalent to (51). Therefore, it follows from Lemma 5 that $P > 0$. This proves that (1) $\Rightarrow$ (3), completing the proof of Lemma 4.

B Details of proof of Theorem 3

Consider the second term in the definition of $V_\Delta(x,t)$:

$$\int_0^t \begin{bmatrix} x(\tau) \\ x_1(\tau) \\ x_2(\tau) \\ p(\tau) \end{bmatrix}^T \begin{bmatrix} D_\psi C_q & C_\psi & 0 & D_\psi D_{qp} \\ 0 & 0 & C_\psi & D_\psi \end{bmatrix}^T \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix} \begin{bmatrix} D_\psi C_q & C_\psi & 0 & D_\psi D_{qp} \\ 0 & 0 & C_\psi & D_\psi \end{bmatrix} \begin{bmatrix} x(\tau) \\ x_1(\tau) \\ x_2(\tau) \\ p(\tau) \end{bmatrix} d\tau$$

$$= \int_0^t \begin{bmatrix} C_\psi x_1(\tau) + D_\psi q(\tau) \\ C_\psi x_2(\tau) + D_\psi p(\tau) \end{bmatrix}^T \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix} \begin{bmatrix} C_\psi x_1(\tau) + D_\psi q(\tau) \\ C_\psi x_2(\tau) + D_\psi p(\tau) \end{bmatrix} d\tau.$$

From the definition (26) of $x_1(t)$ and $x_2(t)$, we have

$$\frac{d}{dt}x_1(t) = A_\psi x_1(t) + B_\psi q(t), \quad \frac{d}{dt}x_2(t) = A_\psi x_2(t) + B_\psi p(t), \quad x_1(0) = x_2(0) = 0.$$

Recall that $(A_\psi, B_\psi, C_\psi, D_\psi)$ is a state-space realization of the system with transfer function

$$\Psi(j\omega) = \begin{bmatrix} \Psi_0(j\omega)^* & \Psi_1(j\omega)^* & \cdots & \Psi_N(j\omega)^* \end{bmatrix}^*.$$

Let $\psi_i(t)$ be the inverse Fourier Transform of $\Psi_i(j\omega)$, $i = 1, \ldots, N$. Then,

$$C_\psi x_1(t) + D_\psi q(t) = \begin{bmatrix} q(t) \\ (\psi_1 \star q)(t) \\ (\psi_2 \star q)(t) \\ \vdots \\ (\psi_N \star q)(t) \end{bmatrix}, \quad C_\psi x_2(t) + D_\psi p(t) = \begin{bmatrix} p(t) \\ (\psi_1 \star p)(t) \\ (\psi_2 \star p)(t) \\ \vdots \\ (\psi_N \star p)(t) \end{bmatrix}.$$

Since $p(t) = (\Delta \star q)(t)$, and since by construction, Δ and ψ_i commute with each other, we have

$$\begin{bmatrix} p(t) \\ (\psi_1 \star p)(t) \\ (\psi_2 \star p)(t) \\ \vdots \\ (\psi_N \star p)(t) \end{bmatrix} = \begin{bmatrix} \Delta & 0 & \cdots & 0 \\ 0 & \Delta & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & \Delta \end{bmatrix} \star \begin{bmatrix} q(t) \\ (\psi_1 \star q)(t) \\ (\psi_2 \star q)(t) \\ \vdots \\ (\psi_N \star q)(t) \end{bmatrix}.$$

For convenience, we let

$$\tilde{q}(t) = \begin{bmatrix} q(t) \\ (\psi_1 \star q)(t) \\ (\psi_2 \star q)(t) \\ \vdots \\ (\psi_N \star q)(t) \end{bmatrix}, \quad \mathcal{D} = \begin{bmatrix} \Delta & 0 & \cdots & 0 \\ 0 & \Delta & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & & \Delta \end{bmatrix}$$

Thus, the second term in the definition of $V_\Delta(x,t)$ can be rewritten as

$$\int_0^t \begin{bmatrix} \tilde{q}(\tau) \\ (\mathcal{D} \star \tilde{q})(\tau) \end{bmatrix}^T \begin{bmatrix} R & 0 \\ 0 & -R \end{bmatrix} \begin{bmatrix} \tilde{q}(\tau) \\ (\mathcal{D} \star \tilde{q})(\tau) \end{bmatrix} d\tau.$$

This further simplifies to

$$\int_0^t \left(\tilde{q}(\tau)^T R\tilde{q}(\tau) - (\mathcal{D} \star \tilde{q})(\tau)^T R(\mathcal{D} \star \tilde{q})(\tau)\right) \, d\tau.$$

$R \in \mathcal{M}$ is positive-definite, and it is easy to establish that its square-root $R^{1/2} \in \mathcal{M}$. Therefore $R^{1/2}$ and $\mathcal{D}$ commute, and the second term in the definition of $V_\Delta(x,t)$ can be rewritten as

$$\int_0^t \left((R^{1/2}\tilde{q}(\tau))^T(R^{1/2}\tilde{q}(\tau)) - (\mathcal{D} \star R^{1/2}\tilde{q})(\tau)^T(\mathcal{D} \star R^{1/2}\tilde{q})(\tau)\right) \, d\tau \geq 0,$$

with the last inequality following from the facts that: (i) $\mathbf{L}_2$ gain of Δ, and consequently that of $\mathcal{D}$, does not exceed one; (ii) the state of Δ is initialized to zero.

Remark 19 The condition that $\mathbf{L}_2$ gain of Δ does not exceed one means that if $p(t) = \Delta(q,t)$, then there exists $\beta \in \mathbf{R}$ such that

$$\forall q \in \mathbf{L}_{2e}, \quad \forall t > 0, \int_0^t p(\tau)^T p(\tau) \, d\tau \leq \int_0^T q(\tau)^T q(\tau) \, d\tau + \beta \tag{56}$$

As we have assumed that the state of Δ is initialized to zero, $\beta = 0$. (For details, see for example, [15, 27, 28].) $\diamond$