Efficient Computation of a Guaranteed Lower Bound on the Robust Stability Margin for a Class of Uncertain Systems*

V. Balakrishnan[†] and F. Wang

Abstract

Sufficient conditions for the robust stability of a class of uncertain systems, with several different assumptions on the structure and nature of the uncertainties, can be derived in a unified manner in the framework of integral quadratic constraints. These sufficient conditions, in turn, can be used to derive lower bounds on the robust stability margin for such systems. The lower bound is typically computed with a bisection scheme, with each iteration requiring the solution of a linear matrix inequality feasibility problem. We show how this bisection can be avoided altogether by reformulating the lower bound computation problem as a single generalized eigenvalue minimization problem, which can be solved very efficiently using standard algorithms. We illustrate this with several important, commonly-encountered special cases: Diagonal, nonlinear uncertainties; diagonal, memoryless, time-invariant sector-bounded ("Popov") uncertainties; structured dynamic uncertainties; and structured parametric uncertainties. We also present a numerical example that demonstrates the computational savings that can be obtained with our approach.

Keywords: Robust stability margin, structured uncertainties, integral quadratic constraints, linear matrix inequalities, generalized eigenvalue minimization.

1 Introduction

Consider the interconnection of a linear system with transfer function $H(s)$ and an uncertainty or perturbation Δ, described by

$$\dot{x} = Ax + Bw, \quad z = Cx + Dw, \quad w = e + \gamma p, \quad p = \Delta q, \quad q = f + z, \tag{1}$$

where $x(t) \in \mathbf{R}^n$, $w(t) \in \mathbf{R}^{n_w}$, $z(t) \in \mathbf{R}^{n_z}$, A, B, C and D are real matrices of appropriate sizes, and $\Delta : \mathbf{L}_2^{n_z}[0,\infty) \to \mathbf{L}_2^{n_w}[0,\infty)$. $\gamma > 0$ is a parameter that serves to "scale" the uncertainty Δ. We assume that all the eigenvalues of A have negative real parts. We also assume that $n_z = n_w = m$; the results herein can be extended with little difficulty to cover the more general cases. Finally, we assume that system (1) is well-posed. The H–Δ interconnection is shown in Fig. 1.

Usually, information about the size of the uncertainty is available. In addition, the uncertainty is often either known or assumed to be diagonal or block-diagonal, sector-bounded memoryless,

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[†]The authors are with the School of Electrical and Computer Engineering, Purdue University, West Lafayette, IN 47907-1285, USA. (e-mail: `ragu@ecn.purdue.edu`).

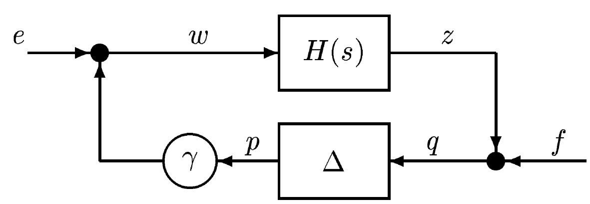

Figure 1: A standard framework for robustness analysis

linear time-invariant (LTI) or parametric, etc. A general framework that encompasses these and many other special cases is that of *Integral Quadratic Constraints* or *IQCs*; see [1] and the references therein. Borrowing notation and terminology from [1], two signals $p \in \mathbf{L}_2^m[0,\infty)$ and $q \in \mathbf{L}_2^m[0,\infty)$, with Fourier Transforms $\hat{p}$ and $\hat{q}$ respectively, are said to "satisfy the IQC defined by Π", if

$$\int_{-\infty}^{\infty} \begin{bmatrix} \hat{q}(j\omega) \\ \hat{p}(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \hat{q}(j\omega) \\ \hat{p}(j\omega) \end{bmatrix} d\omega \geq 0, \tag{2}$$

where $\Pi : j\mathbf{R} \to \mathbf{C}^{2m\times 2m}$ is a measurable Hermitian function, bounded on the imaginary axis. We also say that $\Delta : \mathbf{L}_2^m[0,\infty) \to \mathbf{L}_2^m[0,\infty)$ "satisfies the IQC defined by Π", if for every $q \in \mathbf{L}_2^m[0,\infty)$, q and Δq satisfy the IQC defined by Π.

With the above terminology, we assume that Δ lies in the set

$$\boldsymbol{\Delta} = \{\boldsymbol{\Delta} \mid \text{For every } \Pi \in \boldsymbol{\Pi}, \text{ any } \tau\Delta \text{ satisfies the IQC defined by } \Pi, \text{ where } \tau \in [0,1]\}, \tag{3}$$

where $\boldsymbol{\Pi}$ is some specified set. (We will consider a number of special cases for $\boldsymbol{\Pi}$ in the sequel.) The set $\boldsymbol{\Pi}$ can be thought of as summarizing all the information known about Δ. We will make the following assumption about $\boldsymbol{\Pi}$: Partitioning any $\Pi \in \boldsymbol{\Pi}$ as $\Pi = \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}$, we assume that:

$$\text{For some } \epsilon > 0, \text{ for all } \omega \in \mathbf{R}, \ \Pi_{11}(j\omega) \geq 2\epsilon \text{ and } \Pi_{22}(j\omega) \leq -2\epsilon. \tag{4}$$

We shall see in Section 3 that for a number of commonly encountered uncertainty descriptions, the set $\boldsymbol{\Pi}$ defining the corresponding IQCs satisfies this assumption; thus it is not very restrictive.

Given some $\gamma > 0$, we say that system (1) is robustly stable if it is $\mathbf{L}_2$-stable (see [2]) for every $\Delta \in \boldsymbol{\Delta}$. The quantity of interest, in this paper, is the *robust stability margin* $\gamma_{\rm m}$ of system (1), which is defined as the largest γ such that system (1) is robustly stable. The quantity $\gamma_{\rm m}$ is very useful in practice; for instance, when the uncertainty set $\boldsymbol{\Delta}$ is normalized, $\gamma_{\rm m}$ can be interpreted as the largest uncertainty size for which the H–Δ interconnection in Fig. 1 is robustly stable. It is well-known that computing $\gamma_{\rm m}$ exactly is an NP-hard problem in several important and commonly-encountered situations [3]. (Roughly speaking, this means that the computational effort required to compute $\gamma_{\rm m}$ to within a given accuracy grows more than polynomially with the problem size.) Therefore, we will be content with computing lower bounds on $\gamma_{\rm m}$.

For fixed γ, a number of sufficient conditions for the robust stability of system (1) exist, depending on $\boldsymbol{\Delta}$. When Δ can be *any* operator satisfying an $\mathbf{L}_2$-gain bound, the small-gain theorem provides a necessary and sufficient condition for robust stability. When Δ is structured—say diagonal—the small gain condition is no longer necessary for stability; diagonal scalings can then be used to derive less conservative robust stability conditions [4, 5]. In addition, if Δ is a memoryless time-invariant sector-bounded nonlinearity, the celebrated Popov criterion yields a sufficient condition for robust stability (see for example, [2]). When Δ is LTI or parametric, the well-known μ analysis and K_m analysis methods provide sufficient conditions for robust stability [6, 7, 8]. It has been noted recently that several of these stability criteria can be unified in the setting of stability analysis using IQCs [1]. These stability criteria can be used to define a *guaranteed* lower bound on $\gamma_{\rm m}$ (as the largest γ for which robust stability can be proved using the IQC framework). The computation of the lower bound on $\gamma_{\rm m}$ thus defined requires bisection schemes, with each iteration requiring the solution of a convex feasibility problem, typically a linear matrix inequality (LMI) feasibility problem [23, 9, 10, 11, 12, 13].

The main contribution of this paper is to show how bisection can be avoided altogether, by reformulating the lower bound computation problem as a *single* generalized eigenvalue minimization problem (GEVP)[1]. This is a quasiconvex optimization problem over LMIs, and can be solved very efficiently using standard algorithms and software (see, for example, [14, 15] and [16]). We also present examples that illustrate the computational improvement obtained with our approach.

The organization of the paper is as follows. In Section 2, we very briefly review the robust stability analysis of system (1) using the IQC framework. We then show how to recast the robust stability margin lower bound computation problem as a GEVP. In Section 3, we illustrate our approach on several important commonly-encountered special cases for the set of uncertainties $\mathbf{\Delta}$. In Section 4, we compare the computational effort of the GEVP and bisection schemes with a simple numerical example.

2 Robust stability margin bound via generalized eigenvalue minimization

We review a robust stability criterion, taken from [1], for systems with uncertainties described by IQCs. (For all the uncertainties considered here, this stability criterion can be derived via an application of the passivity theorem with multipliers, see for example, [2, 10, 9, 12].)

Given some $\gamma > 0$, a sufficient condition for the stability of system (1) for all $\Delta \in \mathbf{\Delta}$ is given by the following lemma [1, Theorem 1].

Lemma 1 *Suppose that the interconnection of $H(j\omega)$ and $\tau\Delta$ in Fig. 1 is well-posed for any $\tau \in [0, \gamma]$ and any $\Delta \in \mathbf{\Delta}$. Then, if there exist $\Pi \in \mathbf{\Pi}$ and $\epsilon > 0$ such that*

$$\begin{bmatrix} \gamma H(j\omega) \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \gamma H(j\omega) \\ I \end{bmatrix} \leq -2\epsilon I, \quad \textit{for all } \omega \in \mathbf{R}, \tag{5}$$

system (1) is robustly stable for all $\Delta \in \mathbf{\Delta}$.

The above sufficient condition for the robust stability of system (1) yields a lower bound for the robust stability margin via the following optimization problem:

$$\begin{array}{ll} \texttt{Maximize:} & \gamma \\ \texttt{Subject to:} & \exists \Pi \in \mathbf{\Pi},\ \epsilon > 0,\ \texttt{s.t. (5) holds.} \end{array} \tag{6}$$

We now describe the current, commonly used technique for the numerical solution of Problem (6) (see for example [23]). In general, $\mathbf{\Pi}$—the set defining the IQCs corresponding to $\mathbf{\Delta}$—is not

[1] A similar reduction of the stability margin calculation to a GEVP was also made in [23], however with severe restrictions on the system matrices.

described by a finite number of variables. In order to reduce the number of optimization variables to a finite number, a subset of $\mathbf{\Pi}$ is defined as

$$\mathbf{\Pi}_{\text{fin}} = \left\{ \Pi \;\middle|\; \begin{array}{l} \Pi(j\omega) = \begin{bmatrix} W(j\omega)^* R_{11} W(j\omega) & W(j\omega)^* R_{12} W(j\omega) \\ W(j\omega)^* R_{12}^T W(j\omega) & -W(j\omega)^* R_{22} W(j\omega) \end{bmatrix}, \\ W(j\omega) = \begin{bmatrix} C_W (j\omega I - A_W)^{-1} B_W \\ D_W \end{bmatrix}, \quad \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & -R_{22} \end{bmatrix} \in \Omega, \\ \text{for some } \epsilon > 0, \ \text{ for all } \omega \in \mathbf{R}, \\ \qquad W(j\omega)^* R_{11} W(j\omega) \geq 2\epsilon I, \ W(j\omega)^* R_{22} W(j\omega) \geq 2\epsilon I \end{array} \right\}, \tag{7}$$

where $A_W \in \mathbf{R}^{n_W \times n_W}$, $B_W \in \mathbf{R}^{n_W \times m}$, $C_W \in \mathbf{R}^{K \times n_W}$, $D_W \in \mathbf{R}^{L \times m}$, and Ω is an appropriately chosen subspace of $\mathbf{R}^{2(K+L) \times 2(K+L)}$. (We will see specific examples in Section 3.)

Remark 2 *It is typically computationally more efficient to parameterize the various blocks Π_{ij} of Π in (7) as $\Pi_{ij}(j\omega) = W_i(j\omega)^* R_{ij} W_j(j\omega)$, where W_i and W_j are different transfer functions. However, for simplicity of presentation, we will continue with the definition of $\mathbf{\Pi}_{\text{fin}}$ as in (7), noting that the development herein can be readily extended to the more general case.*

With Π restricted to lie in $\mathbf{\Pi}_{\text{fin}}$, we have another lower bound on the robust stability margin via the following finite-dimensional optimization problem:

$$\begin{array}{ll} \texttt{Maximize:} & \gamma \\ \texttt{Subject to:} & \exists \Pi \in \mathbf{\Pi}_{\text{fin}}, \ \epsilon > 0, \text{ s.t. (5) holds.} \end{array} \tag{8}$$

For a fixed γ, checking if there exists $\Pi \in \mathbf{\Pi}_{\text{fin}}$ such that condition (5) holds can be reformulated as a convex feasibility problem with LMI constraints (see Lemma 4 below). Current techniques take advantage of this observation to solve Problem (8) using a bisection scheme; see [23, 12, 9, 13].

There are a number of problems associated with using the bisection scheme to solve Problem (8). First, upper and lower bounds on the optimal value γ_{opt} need to be determined to initialize the bisection; such bounds may be known *a priori* or may have to be determined. The quality of these bounds will of course affect the efficiency of the bisection scheme in computing γ_{opt}. Moreover, the bisection scheme does not take advantage of the fact that Problem (8) is a quasiconvex optimization problem (see for example, [14] and the references therein).

We now show how the drawbacks with the bisection scheme can be avoided altogether, by reformulating Problem (8) as a Generalized Eigenvalue Minimization Problem or GEVP. A GEVP is an optimization problem of the form

$$\begin{array}{ll} \texttt{Minimize:} & \lambda \\ \texttt{Subject to:} & \lambda B(x) - A(x) > 0, \ \ B(x) > 0, \ \ C(x) > 0. \end{array}$$

Here $x \in \mathbf{R}^p$ is the optimization variable, and $A(x)$, $B(x)$ and $C(x)$ are symmetric matrices that are affine functions of x, i.e.,

$$A(x) = A_0 + x_1 A_1 + \cdots + x_p A_p, \ B(x) = B_0 + x_1 B_1 + \cdots + x_p B_p, \ C(x) = C_0 + x_1 C_1 + \cdots + x_p C_p,$$

where A_i, B_i and C_i are given symmetric matrices. GEVPs are quasiconvex optimization problems based on linear matrix inequalities, and can be solved very efficiently using standard algorithms (see for example, [14, 15, 16]). In particular, as we will demonstrate in Section 4, the solution of Problem (8) as a GEVP leads to considerable computational savings over the solution via a bisection scheme.

The following restatement of the positive-real lemma, taken from [17], plays a central role in the reformulation.

Lemma 3 *Let $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$ and $M = M^T \in \mathbf{R}^{(m+n)\times(m+n)}$, with A having no eigenvalues on the imaginary axis. Then, the following statements are equivalent.*

1. *For some $\epsilon > 0$,*

$$\begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix}^* M \begin{bmatrix} (j\omega I - A)^{-1}B \\ I \end{bmatrix} \geq 2\epsilon I, \qquad \text{for all } \omega \in \mathbf{R}.$$

2. *There exists a symmetric matrix $P = P^T$ such that*

$$\begin{bmatrix} A^TP + PA & PB \\ B^TP & 0 \end{bmatrix} < M.$$

Given $\gamma > 0$, Lemma 3 then enables us to restate the condition that (5) is feasible for some $\Pi \in \mathbf{\Pi}_{\text{fin}}$ as an LMI feasibility condition.

Lemma 4 *Let*

$$\tilde{A} = \begin{bmatrix} A_W & B_WC & 0 \\ 0 & A & 0 \\ 0 & 0 & A_W \end{bmatrix}, \quad \tilde{B} = \begin{bmatrix} B_WD \\ B \\ B_W \end{bmatrix}, \quad \tilde{C} = \begin{bmatrix} I & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix},$$
$$\tilde{D} = \begin{bmatrix} 0 \\ D \\ 0 \\ I \end{bmatrix}, \quad E = \begin{bmatrix} C_W & 0 \\ 0 & D_W \end{bmatrix}. \tag{9}$$

Then, given $\gamma > 0$, there exist some $\Pi \in \mathbf{\Pi}_{\text{fin}}$ and $\epsilon > 0$ such that condition (5) holds if and only if there exist symmetric matrices $P = P^T$, $Q_1 = Q_1^T$ and $Q_2 = Q_2^T$, and $\begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & -R_{22} \end{bmatrix} \in \Omega$ such that

$$M_1 = E^TR_{11}E - \begin{bmatrix} A_W^TQ_1 + Q_1A_W & Q_1B_W \\ B_W^TQ_1 & 0 \end{bmatrix} > 0, \tag{10a}$$

$$M_2 = E^TR_{22}E - \begin{bmatrix} A_W^TQ_2 + Q_2A_W & Q_2B_W \\ B_W^TQ_2 & 0 \end{bmatrix} > 0, \tag{10b}$$

$$\begin{bmatrix} \tilde{A}^TP + P\tilde{A} & P\tilde{B} \\ \tilde{B}^TP & 0 \end{bmatrix} + \begin{bmatrix} \tilde{C}^T \\ \tilde{D}^T \end{bmatrix} \begin{bmatrix} \gamma M_1 & E^TR_{12}E \\ E^TR_{12}^TE & -\frac{1}{\gamma}M_2 \end{bmatrix} \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix} < 0. \tag{10c}$$

Proof: Consider the condition that given $\gamma > 0$, there exist $\Pi \in \mathbf{\Pi}_{\text{fin}}$ and $\epsilon > 0$ such that condition (5) holds, i.e., there exists $\begin{bmatrix} R_{11} & R_{12} \\ R_{12}^T & -R_{22} \end{bmatrix} \in \Omega$ such that with

$$W(j\omega) = \begin{bmatrix} C_W(j\omega I - A_W)^{-1}B_W \\ D_W \end{bmatrix},$$

we have:

$$\text{For some } \epsilon > 0, \text{ for all } \omega \in \mathbf{R}\ , W(j\omega)^* R_{11} W(j\omega) \geq 2\epsilon I \text{ and } W(j\omega)^* R_{22} W(j\omega) \geq 2\epsilon I. \tag{11a}$$

For some $\epsilon > 0$, for all $\omega \in \mathbf{R}$,

$$\begin{bmatrix} H(j\omega) \\ I \end{bmatrix}^* \begin{bmatrix} \gamma W(j\omega)^* R_{11} W(j\omega) & W(j\omega)^* R_{12} W(j\omega) \\ W(j\omega)^* R_{12}^T W(j\omega) & -\frac{1}{\gamma} W(j\omega)^* R_{22} W(j\omega) \end{bmatrix} \begin{bmatrix} H(j\omega) \\ I \end{bmatrix} \leq -2\epsilon I. \tag{11b}$$

Using Lemma 3, it is easily argued that condition (11a) is equivalent to the existence of $Q_1 = Q_1^T$ and $Q_2 = Q_2^T$ such that

$$M_1 = E^T R_{11} E - \begin{bmatrix} A_W^T Q_1 + Q_1 A_W & Q_1 B_W \\ B_W^T Q_1 & 0 \end{bmatrix} > 0,$$

and

$$M_2 = E^T R_{22} E - \begin{bmatrix} A_W^T Q_2 + Q_2 A_W & Q_2 B_W \\ B_W^T Q_2 & 0 \end{bmatrix} > 0,$$

where E is defined in (9); moreover, it is easily verified that for all $\omega \in \mathbf{R}$,

$$\begin{bmatrix} (j\omega I - A_W)^{-1} B_W \\ I \end{bmatrix}^* M_1 \begin{bmatrix} (j\omega I - A_W)^{-1} B_W \\ I \end{bmatrix} = W(j\omega)^* R_{11} W(j\omega),$$

and

$$\begin{bmatrix} (j\omega I - A_W)^{-1} B_W \\ I \end{bmatrix}^* M_2 \begin{bmatrix} (j\omega I - A_W)^{-1} B_W \\ I \end{bmatrix} = W(j\omega)^* R_{22} W(j\omega).$$

Then, with $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{D}$ as given by (9), condition (11b) can be rewritten as:

For some $\epsilon > 0$, for all $\omega \in \mathbf{R}$,

$$\left(\tilde{C}(j\omega I - \tilde{A})^{-1}\tilde{B} + \tilde{D}\right)^* \begin{bmatrix} \gamma M_1 & E^T R_{12} E \\ E^T R_{12}^T E & -\frac{1}{\gamma} M_2 \end{bmatrix} \left(\tilde{C}(j\omega I - \tilde{A})^{-1}\tilde{B} + \tilde{D}\right) \leq -2\epsilon I. \tag{12}$$

Condition (12), using Lemma 3, is equivalent to the existence of $P = P^T$ such that

$$\begin{bmatrix} \tilde{A}^T P + P\tilde{A} & P\tilde{B} \\ \tilde{B}^T P & 0 \end{bmatrix} + \begin{bmatrix} \tilde{C}^T \\ \tilde{D}^T \end{bmatrix} \begin{bmatrix} \gamma M_1 & E^T R_{12} E \\ E^T R_{12}^T E & -\frac{1}{\gamma} M_2 \end{bmatrix} \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix} < 0. \tag{13}$$

This completes the proof. □

Lemma 4 immediately enables the reformulation of Problem (8) as a GEVP, which is the central result of this paper.

Theorem 1 *Let $\kappa_{\rm opt}$ be the optimal value of the GEVP*

$$\begin{aligned}
&\texttt{Minimize:} \quad \kappa \\
&\texttt{Subject to:} \quad P = P^T,\ Q_1 = Q_1^T,\ Q_2 = Q_2^T,\ X = X^T > 0,\ Y = Y^T > 0, \\
&\left[\begin{array}{cc} R_{11} & R_{12} \\ R_{12}^T & -R_{22} \end{array}\right] \in \Omega, \\
&M_1 = E^T R_{11} E - \left[\begin{array}{cc} A_W^T Q_1 + Q_1 A_W & Q_1 B_W \\ B_W^T Q_1 & 0 \end{array}\right] > 0, \\
&M_2 = E^T R_{22} E - \left[\begin{array}{cc} A_W^T Q_2 + Q_2 A_W & Q_2 B_W \\ B_W^T Q_2 & 0 \end{array}\right] > 0, \\
&\left[\begin{array}{cc} \tilde{A}^T P + P\tilde{A} & P\tilde{B} \\ \tilde{B}^T P & 0 \end{array}\right] + \left[\begin{array}{c} \tilde{C}^T \\ \tilde{D}^T \end{array}\right] \left[\begin{array}{cc} X & E^T R_{12} E \\ E^T R_{12}^T E & -Y \end{array}\right] \left[\begin{array}{cc} \tilde{C} & \tilde{D} \end{array}\right] < 0, \\
&\kappa \left[\begin{array}{cc} X & 0 \\ 0 & M_2 \end{array}\right] - \left[\begin{array}{cc} M_1 & 0 \\ 0 & Y \end{array}\right] > 0,
\end{aligned} \qquad (14)$$

where $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, $\tilde{D}$, and E are defined in (9). Then, the optimal value of Problem (8) is $1/\kappa_{\rm opt}$.

Proof: Follows directly from Lemma 4, with the introduction of "slack" variables X and Y and the change of variable $\kappa = 1/\gamma$. □

3 Specified structured uncertainties

With the preliminaries in Section 2, we now consider a number of special cases for $\mathbf{\Delta}$. In each case, the corresponding set $\mathbf{\Pi}$ defining the IQCs satisfies assumption (4), so that the results of Section 2 apply.

3.1 Diagonal nonlinearities

Suppose that

$$\mathbf{\Pi}^{\rm DNL} = \left\{ \Pi \ \middle| \ \Pi(j\omega) = \left[\begin{array}{cc} W & 0 \\ 0 & -W \end{array}\right],\ W \in \mathbf{R}^{m\times m},\ W > 0 \text{ and diagonal} \right\}. \qquad (15)$$

Every "diagonal" Δ with an $\mathbf{L}_2$ gain that does not exceed one, satisfies every IQC from $\mathbf{\Pi}^{\rm DNL}$. Note that $\mathbf{\Pi}^{\rm DNL}$ is already described by a finite number of variables so that $\mathbf{\Pi}^{\rm DNL} = \mathbf{\Pi}^{\rm DNL}_{\rm fin}$ and is defined by (7), where A_W, B_W and C_W are vacuous, $D_W = I$, and $\Omega = \mathbf{\Pi}^{\rm DNL}$.

From Theorem 1, Problem (8) is equivalent to the GEVP

`Minimize:` κ

`Subject to:` $P = P^T,\ X = X^T > 0,\ Y = Y^T > 0,\ W \in \mathbf{R}^{m\times m}$ and diagonal, $W > 0$,

$$\begin{bmatrix} A^TP + PA & PB \\ B^TP & 0 \end{bmatrix} + \begin{bmatrix} C^T & 0 \\ D^T & I \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \begin{bmatrix} C & D \\ 0 & I \end{bmatrix} < 0, \tag{16}$$

$$\kappa \begin{bmatrix} X & 0 \\ 0 & W \end{bmatrix} - \begin{bmatrix} W & 0 \\ 0 & Y \end{bmatrix} > 0,$$

with $1/\kappa_{\mathrm{opt}}$ being the optimal lower bound on γ_{m}. This GEVP formulation is exactly the same one derived in [18] for minimizing the $\mathbf{H}_\infty$ norm of $H(s) = C(sI - A)^{-1}B + D$ over diagonal similarity scalings (also see [19]).

3.2 Diagonal, memoryless, time-invariant, sector-bounded nonlinearities

Suppose that $D = 0$, that is, H is strictly proper, and

$$\mathbf{\Pi}^{\mathrm{Popov}} = \left\{ \Pi \;\middle|\; \Pi(j\omega) = \begin{bmatrix} \Lambda & -j\omega\Gamma \\ j\omega\Gamma & -\Lambda \end{bmatrix},\ \Lambda, \Gamma \in \mathbf{R}^{m\times m} \text{ and diagonal, } \Lambda > 0 \right\}.$$

The set of uncertainties satisfying every IQC from $\mathbf{\Pi}^{\mathrm{Popov}}$ contains the set of all diagonal, memoryless, time-invariant nonlinearities, with the diagonal entries in sector $[-1, 1]$; see [2, 1]. These are often referred to as diagonal "Popov-type" uncertainties.

Note that the elements of $\mathbf{\Pi}_{\mathrm{fin}}^{\mathrm{Popov}}$ are not bounded on the imaginary axis. But a simple change of variables enables us to address this problem. (This is a standard technique, used to derive the Popov criterion; see [2, 1].) We simply define $\tilde{H}(s) = H(s)(1 + s) = (C + CA)(sI - A)^{-1}B + CB$, and $\tilde{\Delta} = \Delta \circ 1/(1 + s)$, and observe that the stability of the $\tilde{H}$–$\tilde{\Delta}$ interconnection is equivalent to that of the H–Δ interconnection. Now, $\tilde{\Delta}$ satisfies every IQC from

$$\tilde{\mathbf{\Pi}}^{\mathrm{Popov}} = \left\{ \tilde{\Pi} \;\middle|\; \tilde{\Pi}(j\omega) = \begin{bmatrix} \dfrac{1}{1+\omega^2}\Lambda & \dfrac{-j\omega}{1-j\omega}\Gamma \\ \dfrac{j\omega}{1+j\omega}\Gamma & -\Lambda \end{bmatrix},\ \Lambda, \Gamma \in \mathbf{R}^{m\times m} \text{ and diagonal, } \Lambda > 0 \right\}.$$

The set $\tilde{\mathbf{\Pi}}^{\mathrm{Popov}}$ is described by a finite number of variables so that $\tilde{\mathbf{\Pi}}^{\mathrm{Popov}} = \tilde{\mathbf{\Pi}}_{\mathrm{fin}}^{\mathrm{Popov}}$, and is defined by (7) with $A_W = -I$, $B_W = C_W = D_W = I$, and

$$\Omega^{\mathrm{Popov}} = \left\{ \begin{bmatrix} \Lambda & 0 & 0 & -\Gamma \\ 0 & 0 & 0 & \Gamma \\ 0 & 0 & 0 & 0 \\ -\Gamma & \Gamma & 0 & -\Lambda \end{bmatrix} \;\middle|\; \Lambda, \Gamma \in \mathbf{R}^{m\times m} \text{ and diagonal, } \Lambda > 0 \right\}.$$

Then, applying Theorem 1 to the $\tilde{H}$–$\tilde{\Delta}$ interconnection, we get the following GEVP.

Minimize: κ

Subject to: $P = P^T,\ X = X^T > 0,\ Y = Y^T > 0,\ \Lambda$ and Γ are diagonal, $\Lambda > 0$,

$$\begin{bmatrix} \tilde{A}^T P + P\tilde{A} & P\tilde{B} \\ \tilde{B}^T P & 0 \end{bmatrix} + \begin{bmatrix} \tilde{C}^T \\ \tilde{D}^T \end{bmatrix} \begin{bmatrix} X & 0 & 0 & -\Gamma \\ 0 & 0 & 0 & \Gamma \\ 0 & 0 & 0 & 0 \\ -\Gamma & \Gamma & 0 & -Y \end{bmatrix} \begin{bmatrix} \tilde{C} & \tilde{D} \end{bmatrix} < 0,$$

$$\kappa \begin{bmatrix} X & 0 \\ 0 & \Lambda \end{bmatrix} - \begin{bmatrix} \Lambda & 0 \\ 0 & Y \end{bmatrix} > 0,$$

where

$$\tilde{A} = \begin{bmatrix} -I & C + CA & 0 \\ 0 & A & 0 \\ 0 & 0 & -I \end{bmatrix},\ \tilde{B} = \begin{bmatrix} CB \\ B \\ I \end{bmatrix},\ \tilde{C} = \begin{bmatrix} I & 0 & 0 \\ 0 & C + CA & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix},\ \tilde{D} = \begin{bmatrix} 0 \\ CB \\ 0 \\ I \end{bmatrix}.$$

The optimal lower bound on $\gamma_{\rm m}$ is given by $1/\kappa_{\rm opt}$.

3.3 Parametric uncertainties

Suppose Δ is a constant real matrix with a specified block-diagonal structure, and with a spectral norm that does not exceed one: $\Delta = \mathbf{diag}\,(D_1, \ldots, D_M, d_1 I_{\ell_1}, \ldots, d_N I_{\ell_N})$, $D_i \in \mathbf{R}^{k_i \times k_i}$, $i = 1, \ldots, M$, $d_i \in \mathbf{R}$, $i = 1, \ldots, N$, with $\sigma_{\max}(\Delta) \leq 1$. (Note that $\sum k_i + \sum \ell_i = m$.)

Then, Δ satisfies every IQC from

$$\mathbf{\Pi}^{\rm par} = \left\{ \Pi \;\middle|\; \begin{array}{l} \Pi(j\omega) = \begin{bmatrix} X(j\omega) & Y(j\omega) \\ -Y(j\omega) & -X(j\omega) \end{bmatrix} = \Pi(j\omega)^*,\ X(j\omega) \text{ and } Y(j\omega) \in \mathcal{W}, \\ \text{for some } \epsilon > 0,\ X(j\omega) \geq 2\epsilon I,\ Y(j\omega) = -Y^*(j\omega), \text{ for all } \omega \in \mathbf{R} \end{array} \right\},$$

where

$$\mathcal{W} = \left\{ \mathbf{diag}\,(w_1 I_{k_1}, \ldots, w_M I_{k_M}, W_1, \ldots, W_N) \;\middle|\; \begin{array}{l} w_i \in \mathbf{C},\ i = 1, \ldots, M \\ W_i \in \mathbf{C}^{\ell_i \times \ell_i},\ i = 1, \ldots, N \end{array} \right\}. \qquad (17)$$

(See for example [12, 1].)

For such uncertainties, Theorem 1 can be immediately used to obtain GEVPs that yield a guaranteed lower bound on the robust stability margin. Let $\mathcal{P}$ denote the subset of real matrices that lie in $\mathcal{W}$, i.e., $\mathcal{P} = \mathcal{W} \bigcap \mathbf{R}^{m \times m}$.

Then, a subset $\mathbf{\Pi}^{\rm par}_{\rm fin}$ of $\mathbf{\Pi}^{\rm par}$, described by a finite number of variables, can be defined as follows. Let $W^{(1)}, \ldots, W^{(N-1)}$ be strictly proper, stable $m \times m$ transfer functions, with each $W^{(i)}$ satisfying $W^{(i)}(j\omega) \in \mathcal{W}$ for every $\omega \in \mathbf{R}$. Let

$$\mathbf{\Theta}^{\rm par} = \left\{ \begin{bmatrix} \theta_{11} & \theta_{12} & \cdots & \theta_{1N} \\ \theta_{21} & \theta_{22} & \cdots & \theta_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ \theta_{N1} & \theta_{N2} & \cdots & \theta_{NN} \end{bmatrix} \;\middle|\; \theta_{ij} \in \mathcal{P} \right\}. \qquad (18)$$

Then, a subset $\mathbf{\Pi}_{\text{fin}}^{\text{par}}$ of $\mathbf{\Pi}^{\text{par}}$ described by a finite number of variables is given by (7), where (A_W, B_W, C_W) is any state space realization of $\begin{bmatrix} W^{(1)}(s)^T & \cdots & W^{(N-1)}(s)^T \end{bmatrix}^T$, $D_W = I$, and

$$\Omega^{\text{par}} = \left\{ \begin{bmatrix} \Theta + \Theta^T & \Phi - \Phi^T \\ \Phi^T - \Phi & -\left(\Theta + \Theta^T\right) \end{bmatrix} \middle| \; \Theta, \Phi \in \mathbf{\Theta}^{\text{par}} \right\}. \tag{19}$$

Thus, the problem of computing lower bounds on the robust stability margin of systems with parametric uncertainties can be efficiently solved as a GEVP.

Note that the choice of $W^{(i)}$ is *ad hoc*, and the value of κ_{opt} will certainly depend on this choice. However, for any choice of $W^{(i)}$, the inverse of the optimal value κ_{opt} obtained from GEVP (14) is a *guaranteed* lower bound on the robust stability margin γ_{m}. Moreover, it can be shown (see [20]) that the actual choice of the $W^{(i)}$ is immaterial, provided the set of $W^{(i)}$s is chosen to be "rich enough".

3.4 Structured dynamic uncertainties

Suppose Δ is a dynamic block-structured uncertainty with an $\mathbf{L}_2$-gain that does not exceed one (i.e., it is nonexpansive): For all $\omega \in \mathbf{R}$, $\Delta(j\omega) = \mathbf{diag}\,(D_1, \ldots, D_M, d_1 I_{\ell_1}, \ldots, d_N I_{\ell_N})$, $D_i \in \mathbf{C}^{k_i \times k_i}$, $i = 1, \ldots, M$, $d_i \in \mathbf{C}$, $i = 1, \ldots, N$, with $\sigma_{\max}(\Delta(j\omega)) \leq 1$. (Note that $\sum k_i + \sum \ell_i = m$.)

Then, Δ satisfies every IQC from

$$\mathbf{\Pi}^{\text{LTI}} = \left\{ \Pi \;\middle|\; \begin{array}{l} \Pi(j\omega) = \begin{bmatrix} X(j\omega) & 0 \\ 0 & -X(j\omega) \end{bmatrix}, \; X(j\omega) = X(j\omega)^* \in \mathcal{W}, \\ \text{for some } \epsilon > 0, \; X(j\omega) \geq 2\epsilon I, \text{ for all } \omega \in \mathbf{R} \end{array} \right\},$$

where $\mathcal{W}$ is defined in (17) (see for example [12, 1]). Similarly to the development in Section 3.3, $\mathbf{\Pi}_{\text{fin}}^{\text{LTI}} \subset \mathbf{\Pi}^{\text{LTI}}$ can be described by a finite number of variables. Let $W^{(1)}, \ldots, W^{(N-1)}$ be strictly proper, stable $m \times m$ transfer functions, with each $W^{(i)}$ satisfying $W^{(i)}(j\omega) \in \mathcal{W}$ for every $\omega \in \mathbf{R}$. Then, a subset $\mathbf{\Pi}_{\text{fin}}^{\text{LTI}}$ of $\mathbf{\Pi}^{\text{LTI}}$ described by a finite number of variables is given by (7), where (A_W, B_W, C_W) is any state space realization of $\begin{bmatrix} W^{(1)}(s)^T & \cdots & W^{(N-1)}(s)^T \end{bmatrix}^T$, $D_W = I$, and

$$\Omega^{\text{LTI}} = \left\{ \begin{bmatrix} \Theta + \Theta^T & 0 \\ 0 & -\left(\Theta + \Theta^T\right) \end{bmatrix} \middle| \; \Theta \in \mathbf{\Theta}^{\text{par}} \right\}. \tag{20}$$

Thus, the problem of computing lower bounds on the robust stability margin of systems with structured dynamic uncertainties can be efficiently solved as a GEVP.

4 A numerical example

We present an application of the results of this paper on a simple example. Consider an instance of the H–Δ interconnection system with

$$H(s) = \begin{bmatrix} \dfrac{0.2}{s^2+0.1s+0.7} & \dfrac{-1.5}{s^2+0.1s+0.7} & \dfrac{-s^2+0.9s-0.2}{s^3+0.4s^2+0.73s+0.21} \\ \dfrac{s}{s^2+0.1s+0.7} & \dfrac{-7.5s}{s^2+0.1s+0.7} & \dfrac{10s^2+3s+3.5}{s^3+0.4s^2+0.73s+0.21} \\ 0 & 0 & \dfrac{-2}{s+0.3} \end{bmatrix}.$$

With Δ assumed to be diagonal in addition to satisfying various IQCs, we now demonstrate that significant computational savings accrue when the lower bound on the robust stability margin is computed using the GEVP formulation from Theorem 1, as compared to a bisection scheme.

In implementing a bisection scheme to solve Problem (8), we used $\kappa = 1/\gamma$ as the optimization variable. Upper and lower bounds on the optimal value κ_{opt} that are required to initialize the bisection can be computed using different methods; and the performance of the bisection scheme can be made arbitrarily poor by choosing the bounds to be far enough apart. We avoided introducing any such bias against the bisection scheme as follows. In all cases that we consider, Δ satisfies the IQC with $\Pi = \mathbf{diag}(I, -I)$, and therefore $\|H\|_\infty$ is an upper bound on κ_{opt}; this can be computed very efficiently using the algorithms in [21]. A lower bound on κ_{opt} is simply zero. The upper bound can also be readily incorporated into the GEVP (14). We denote the GEVP with this additional linear constraint as GEVPWB.

Table 1 shows a comparison of the performance of the bisection and the GEVP schemes. In every case, κ_{opt} was computed to an relative accuracy of 1%. In the case of diagonal parametric uncertainties, $\mathbf{\Pi}_{\text{fin}}^{\text{par}}$ was given by (7), (18) and (19), with the following choices: $A_W = \mathbf{diag}(-10, -10, -10)$, $B_W = C_W = D_W = I$, and

$$\mathbf{\Theta}^{\text{par}} = \left\{ \begin{bmatrix} \theta_{11} & \theta_{12} \\ \theta_{12} & \theta_{22} \end{bmatrix} \middle| \ \theta_{ij} \in \mathbf{R}^{3\times 3} \text{ and diagonal} \right\}.$$

In the diagonal dynamic uncertainties, $\mathbf{\Pi}_{\text{fin}}^{\text{LTI}}$ was given similarly, with the only difference being in the definition of Ω, which was given by (20). All LMI computations were performed using the LMI Toolbox of MATLAB [16], and computation times on a Sparc 20 are reported here[2]. The numerical results show that GEVP is always considerably more efficient than the bisection scheme. In addition, this example suggests that *a priori* knowledge of an upper bound on κ_{opt} makes little difference to the performance of the GEVP.

[2]Though the values of the time themselves are not very meaningful as they depend on the hardware used, the *ratio* of the computation time of the bisection scheme to the GEVP scheme still provides a meaningful basis for comparison.

Uncertainty Type	$1/\kappa_{\mathrm{opt}}$	Bisection (sec)	GEVP (sec)	GEVPWB (sec)
General nonlinear	1.2896×10^{-2}	5.6800	0.8200	0.8000
Dynamic	1.2899×10^{-2}	62.5700	20.3000	18.9200
Popov-type	1.3264×10^{-2}	99.5700	26.5300	26.1200
Parametric	1.3278×10^{-2}	51.8400	19.8300	19.1700

Table 1: A comparison of the bisection and GEVP schemes. All uncertainties are assumed to be diagonal.

5 Conclusion

We have shown that a guaranteed lower bound on the robust stability margin for a number of commonly encountered uncertain systems can be computed via generalized eigenvalue minimization. Examples show that the GEVP reformulation of the robust stability margin lower bound computation leads to considerable computational savings. The results presented herein also apply to many other uncertain systems, besides the special cases considered in Section 3; see for example, [1] and [22].

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