Robust Stability and Performance Analysis of Uncertain Systems Using Linear Matrix Inequalities[1,2]

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Abstract. A wide variety of problems in system and control theory can be formulated (or reformulated) as convex optimization problems involving linear matrix inequalities (LMIs), that is, constraints requiring an affine combination of symmetric matrices to be positive semidefinite. For a few very special cases, there are "analytical solutions" to these problems, but in general they can be solved numerically very efficiently. Thus, the reduction of a control problem to an optimization problem based on LMIs constitutes, in a sense, a "solution" to the original problem. The objective of this article is to provide a tutorial on the application of optimization based on LMIs to robust control problems. In the first part of the article, we provide a brief introduction to optimization based on LMIs. In the second part, we describe a specific example, that of robust stability and performance analysis of uncertain systems using LMI optimization.

Keywords: Robust stability, robust performance, convex optimization, linear matrix inequalities, Lyapunov functions.

1 Introduction

A wide variety of problems in system and control theory can be reduced to a handful of standard convex and quasiconvex optimization problems that involve linear matrix inequalities or LMIs, that is constraints of the form

$$F(x) \stackrel{\Delta}{=} F_0 + \sum_{i=1}^{m} x_i F_i > 0, \tag{1}$$

where $x \in \mathbf{R}^m$ is the variable, and $F_i = F_i^T \in \mathbf{R}^{n\times n}$, $i = 0, \ldots, m$, are given. Though the form of the LMI (1) appears very specialized, it turns out that it is widely encountered in system and control theory. Examples include: multicriterion LQG, synthesis of linear state feedback for multiple or nonlinear plants ("multi-model control"), optimal state-space realizations of transfer matrices, norm scaling, synthesis of multipliers for Popov-like analysis of systems with unknown gains, robustness analysis and robust controller design, gain-scheduled controller design, and many others. For a few very special cases there are "analytical solutions" to LMI optimization problems, but in general they can be solved numerically very efficiently. Indeed, the recent and growing popularity of LMI optimization for control can be directly traced to the recent breakthroughs in interior point methods for LMI optimization (see for example, Refs. 1–4). In many cases—for example, with multi-model control (Ref. 5)—the LMIs encountered in systems and control theory have the form of simultaneous (coupled) Lyapunov or algebraic Riccati inequalities; using interior-point methods, such problems can be solved in a time that is roughly comparable to the time required to solve the same number of (uncoupled) Lyapunov or Algebraic Riccati equations (Ref. 6). Therefore the computational cost of extending current control theory that is based on the solution of algebraic Riccati equations to a theory based on the solution of (multiple, simultaneous) Lyapunov or Riccati inequalities is modest.

A number of publications can be found in the control literature that survey applications of LMI optimization to the solution of system and control problems. Perhaps the most comprehensive list can be found in the book Ref. 5. Since its publication, a number of papers have appeared chronicling further applications of LMI optimization techniques in control; a few examples are Refs. 7 and 8. The growing popularity of LMI methods for control is also evidenced by the large number of publications in recent control conferences.

Our first objective in this paper is to give a brief introduction to optimization based on LMIs. In Section 2.1, we describe a few "standard" convex and quasiconvex optimization problems involving LMIs. We make a few brief remarks about solving LMI-based optimization problems in Section 2.2. In Section 2.3, we present a brief history of LMIs in system and control theory.

Our second objective is to present one specific example of the application of LMI optimization for control. The control problem that we consider, in Sections 3–5, is that of robust stability and performance analysis of uncertain systems, with various assumptions on the nature of the uncertainties (sector-bounded nonlinear, linear time-invariant, parametric, etc.), as well as their structure (diagonal, block-diagonal, etc.). We first show, in Section 4, how the robust stability analysis of such systems can be performed in a unified manner using multiplier theory and LMI-based convex optimization. Not only does this provide a unification of several apparently-diverse robust stability tests, but it also paves the way for developing new stability tests. In addition, the multipliers used in the stability analysis can be shown to yield a convex parametrization of a subset of Lyapunov functions that provide a certificate of robust stability. In Section 5, we show how these Lyapunov functions can in turn be used to derive bounds on various robust performance measures for uncertain systems. We illustrate our approach with two specific robust performance analysis problems.

2 Optimization Based on Linear Matrix Inequalities

Recall the definition of a linear matrix inequality:

$$F(x) \stackrel{\Delta}{=} F_0 + \sum_{i=1}^{m} x_i F_i > 0,$$

where $x \in \mathbf{R}^m$ is the variable, and $F_i = F_i^T \in \mathbf{R}^{n \times n}$, $i = 0, \ldots, m$ are given. The set $\{x \mid F(x) > 0\}$ is convex, and need not have smooth boundary. (We have used strict inequality mostly for convenience; inequalities of the form $F(x) \geq 0$ are also readily handled.)

Multiple LMIs $F_1(x) > 0, \ldots, F_n(x) > 0$ can be expressed as the single LMI

$$\mathbf{diag}\,(F_1(x), \ldots, F_n(x)) > 0.$$

When the matrices F_i are diagonal, the LMI $F(x) > 0$ is just a set of linear inequalities. Nonlinear (convex) inequalities are converted to LMI form using Schur complements. The basic idea is as follows: the LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0, \tag{2}$$

where $Q(x) = Q(x)^T$, $R(x) = R(x)^T$, and $S(x)$ depend affinely on x, is equivalent to

$$R(x) > 0, \; Q(x) - S(x)R(x)^{-1}S(x)^T > 0. \tag{3}$$

In other words, the set of nonlinear inequalities (3) can be represented as the LMI (2).

The matrix norm constraint $\|Z(x)\| < 1$, where $Z(x) \in \mathbf{R}^{p \times q}$ and depends affinely on x, is represented as the LMI

$$\begin{bmatrix} I & Z(x) \\ Z(x)^T & I \end{bmatrix} > 0$$

(since $\|Z\| < 1$ is equivalent to $I - ZZ^T > 0$). Note that the case $q = 1$ reduces to a general convex quadratic inequality on x.

The constraint

$$\mathbf{Tr}\, S(x)^T P(x)^{-1} S(x) < 1, \quad P(x) > 0,$$

where $P(x) = P(x)^T \in \mathbf{R}^{n\times n}$ and $S(x) \in \mathbf{R}^{n\times p}$ depend affinely on x, is handled by introducing a new (slack) matrix variable $X = X^T \in \mathbf{R}^{p\times p}$, and the LMI (in x and X):

$$\mathbf{Tr}\, X < 1, \quad \begin{bmatrix} X & S(x)^T \\ S(x) & P(x) \end{bmatrix} > 0.$$

We often encounter problems in which the variables are matrices, e.g.,

$$A^T P + PA < 0, \tag{4}$$

where $A \in \mathbf{R}^{n\times n}$ is given and $P = P^T$ is the variable. In this case we will not write out the LMI explicitly in the form $F(x) > 0$, but instead make clear which matrices are the variables. Leaving LMIs in a condensed form such as (4), in addition to saving notation, leaves open the possibility of more efficient computation.

2.1 Some Standard LMI Optimization Problems

(i) LMI Feasibility Problems

Given an LMI $F(x) > 0$, the corresponding LMI feasibility problem is to find $x^{\rm feas}$ such that $F(x^{\rm feas}) > 0$ or determine that the LMI is infeasible. (By duality, this means: find a nonzero $G \geq 0$ such that $\mathbf{Tr}\, GF_i = 0$ for $i = 1, \ldots, m$ and $\mathbf{Tr}\, GF_0 \leq 0$.) Of course, this is a convex feasibility problem.

(ii) Eigenvalue Problems

The eigenvalue problem is to minimize the maximum eigenvalue of a matrix, subject to an LMI:

$$\begin{array}{ll} \min & \lambda \\ \\ \text{s.t.} & \lambda I - A(x) > 0, \quad B(x) > 0. \end{array}$$

Here, A and B are symmetric matrices that depend affinely on the optimization variable x. This is a convex optimization problem.

(iii) Generalized Eigenvalue Problems

The generalized eigenvalue problem is to minimize the maximum generalized eigenvalue of a pair of matrices that depend affinely on the optimization variable, subject to an LMI constraint. The general form of a generalized eigenvalue problem is:

$$\begin{array}{ll} \min & \lambda \\ \\ \text{s.t.} & \lambda B(x) - A(x) > 0 \\ & B(x) > 0 \\ & C(x) > 0 \end{array}$$

where A, B and C are affine functions of x. This is a quasiconvex problem.

Note that when the matrices are all diagonal, this problem reduces to the general linear fractional programming problem. Many nonlinear quasiconvex functions can be represented in the form of a generalized eigenvalue problem with appropriate A, B, and C (see Ref. 9).

2.2 Solving LMI-Based Problems

The most important point is:

LMI feasibility, eigenvalue and generalized eigenvalue problems are all tractable

in a sense that can be made precise from a number of theoretical and practical viewpoints. (This is to be contrasted with much less tractable problems, e.g., the general problem of robustness analysis for a system with real parameter perturbations.)

From a theoretical standpoint:

- We can immediately write down necessary and sufficient optimality conditions.
- There is a well-developed duality theory (for generalized eigenvalue problems, in a limited sense).
- These problems can be solved in polynomial time (indeed with a variety of interpretations of the term "polynomial-time").

The most important practical implication is that there are effective and powerful algorithms for the solution of these problems, that is, algorithms that rapidly compute the global optimum, with non-heuristic stopping criteria. Thus, on exit, the algorithms can prove that the global optimum has been obtained to within some prespecified accuracy.

There are a number of general algorithms for the solution of LMI problems, for example, the ellipsoid algorithm (see e.g., Refs. 10 and 11). The ellipsoid method has polynomial-time complexity, and works in practice for smaller problems, but can be slow for larger problems. Other algorithms specifically for LMI-based problems are discussed in, e.g., Refs. 12 and 13. More recently, various researchers (Refs. 2–4) have developed interior point methods for solving LMI-based problems, based on the work of Nesterov and Nemirovskii (Ref. 1). Numerical experience shows that these algorithms solve LMI problems with great efficiency. A survey of algorithms and software for LMI optimization can be found in Ref. 8.

A number of software packages are also available for solving LMI problems. The first implementation of an interior-point method for LMI problems was by Nesterov and Nemirovskii in Ref. 14, using the projective algorithm (Ref. 1). Matlab's LMI Control Toolbox (Ref. 15) is based on the same algorithm, and offers a graphical user interface and extensive support for control applications. The code SP (Ref. 16) is based on a primal-dual potential reduction method with the Nesterov and Todd scaling. The code is written in C with calls to BLAS and LAPACK and includes an interface to Matlab. SDPSOL (Ref. 17) and LMITOOL (Ref. 18) offer user-friendly interfaces to SP that simplify the specification of LMI problems where the variables have matrix structure. The Induced-Norm Control Toolbox (Ref. 19) is a toolbox for robust and optimal control, in turn based on LMITOOL.

2.3 Brief History of LMIs in System and Control Theory

The history of linear matrix inequalities in the analysis of dynamical systems goes back more than 100 years, when Lyapunov published his seminal work introducing what we now call Lyapunov theory. He showed that the differential equation

$$\frac{d}{dt}x(t) = Ax(t) \tag{5}$$

is stable if and only if there exists a positive definite matrix P such that

$$A^T P + PA < 0. \tag{6}$$

The requirement $P > 0$, $A^T P + PA < 0$ is what we now call a Lyapunov inequality on P, which is a special form of an LMI. Of course, we can solve this LMI (that is, find a suitable P) by solving a Lyapunov equation.

The next major development was in the 1940s when Lur'e, Postnikov, and others in the Soviet Union applied Lyapunov's methods to some specific practical problems in control engineering, especially, the problem of stability of a control system with a nonlinearity in the actuator (Ref. 20). Although they did not explicitly form matrix inequalities, their stability criteria in fact have the form of LMIs. These inequalities were reduced to polynomial inequalities which were then checked "by hand" (for, needless to say, small systems).

Then, in the 1960s, Yakubovich, Popov, Kalman, and other researchers succeeded in reducing the solution of the LMIs that arose in the problem of Lur'e to simple graphical criteria, using what we now call the Kalman-Yakubovich-Popov (KYP) lemma. This resulted in the celebrated Popov criterion, Circle criterion, Tsypkin criterion, and many variations. These criteria could be applied to higher order systems, but did not gracefully or usefully extend to systems containing more than one nonlinearity. Thus, their contribution may be viewed—in the context of the history of LMIs in control theory—as showing how to solve a certain family of LMIs by a graphical method. We should note that the important role of LMIs in control theory was already recognized in the early 1960s, especially by Yakubovich (Ref. 21).

By 1971, researchers knew several methods for solving special types of LMIs: direct (for very small systems), graphical methods, and by solving Lyapunov or Riccati equations. In Willems' 1971 paper (Ref. 22) we find the following striking quote:

> The basic importance of the LMI seems to be largely unappreciated. It would be interesting to see whether or not it can be exploited in computational algorithms, for example.

Willems' suggestion that LMIs might have some advantages in computational algorithms became closer to being realized with the following observation:

> The LMIs that arise in system and control theory can be formulated as *convex optimization problems*, and hence are amenable to computer solution.

This observation was made explicitly by several researchers: Pyatnitskii and Skorodinskii (Ref. 23), and Horisberger and Belanger (Ref. 24), to name just a few. This observation, coupled with the development of interior point methods that apply directly to convex problems involving matrix inequalities, by Nesterov and Nemirovskii in 1988, now mean that we can reliably and quickly solve many problems in systems and control for which no "analytical solution" has been found (or is likely to be found), by reducing to them to LMI problems. The efficient solution of LMI optimization problems as well as the numerical solution of several diverse engineering problems via LMI optimization remain areas of active research.

3 Robust Stability and Performance Analysis of Uncertain Systems: Overview

Most control system models in use today explicitly incorporate in them "uncertainties" or "perturbations". These uncertainties may model a number of factors, including: dynamics that are neglected to make the model tractable, as with large scale structures; nonlinearities that are either hard to model or too complicated; and parameters that are not known exactly, either because they are hard to measure or because of varying manufacturing conditions. A widely-used model for uncertain systems, shown in Fig. 1, consists of a nominal finite-dimensional, stable, linear time-invariant system, with a perturbation or uncertainty Δ in a feedback loop. The signal w represents exogenous inputs, and z represents all outputs of interest. Often additional information about Δ is either known or assumed; common examples are that Δ is diagonal or block-diagonal; sector-bounded memoryless, linear time-invariant or parametric; bounded in norm, passive, etc. The analysis of and design for such control system models is commonly referred to as "robust control with structured perturbations".

One of the most fundamental questions concerning the system in Fig. 1 is that of stability: "Is the model stable irrespective of the perturbation Δ, that is, do all solutions of the system equations go to zero, irrespective of Δ?" This is also referred to as the *robust stability* problem. Some of the approaches for solving this problem, with various assumptions on Δ, are the use of the small-gain, passivity or circle-criteria (Ref. 25), the Popov criterion (Ref. 26), and μ or K_m analysis methods (Refs. 27–30). Robust stability is but one desired feature of an uncertain system; of considerable importance are questions beyond stability, known broadly as *robust performance* problems. Robust performance analysis problems concern the computation of the worst possible value, over all uncertainties, of performance indices; these performance indices may be bounds on some state variables, norms of the map from w to z etc. An example of a robust performance analysis problem is the so-called $\mathbf{H}_2$ problem, which is the computation of the largest possible RMS value of the output z, over all Δ, when the input w is unit-intensity white noise. This finds application where the average value of a certain signal is of interest, when the system is affected by an unpredictable input that is modeled as white noise. Another example is the computation of the largest possible RMS gain, over all uncertainties, from w to z; this is also known as the $\mathbf{H}_\infty$ performance analysis problem.

It has been found recently that several stability analysis methods for control systems can be unified in the setting of multiplier theory (Refs. 25 and 31–34), or more generally, in the framework of integral quadratic constraints or IQCs (Ref. 35). (While the framework of IQCs is more general than the multipier-based framework, we have chosen to present the latter here for "historical" reasons.) As a consequence, several sufficient conditions for robust stability can be performed without frequency sampling, using convex optimization techniques based on LMIs (Ref. 33); we describe this in Section 4. It also turns out that the multiplier techniques yield a convex parametrization of a set of Lyapunov functions that prove robust stability (Ref. 36). By imposing additional conditions on these Lyapunov functions, bounds on the robust performance for the system in Fig. 1 can be derived; thus Lyapunov functions can be used to "prove" robust performance as well. The "best" performance bounds can then be obtained by numerically optimizing these bounds, using LMI-based methods, over the set of Lyapunov functions that prove robust performance. We describe this approach in Section 5.

4 Robust Stability Using Multiplier Theory

Let the equations governing the system in Fig. 1 be:

$$\frac{d}{dt}x(t) = Ax(t) + B_p p(t) + B_w w(t), \quad q(t) = C_q x(t) + D_{qp} p(t), \quad z(t) = C_z x(t), \tag{7a}$$

$$p(t) = -\Delta(q,t), \tag{7b}$$

where $x(t) \in \mathbf{R}^n$, $p(t) \in \mathbf{R}^{n_p}$, $q(t) \in \mathbf{R}^{n_q}$, $w(t) \in \mathbf{R}^{n_w}$ and $z(t) \in \mathbf{R}^{n_z}$. For convenience, we have assumed that there is no feed-through from w to z, w to q and p to z; we will also assume that $n_p = n_q = m$. For future reference, we let $\mathcal{L}$ denote the linear system (7a), and G the transfer function of the linear part of the system from p to q, i.e., $G(s) = C_q(sI - A)^{-1}B_p + D_{qp}$. $\mathcal{L}$ is assumed to be stable. Equations (7) can be interpreted either as representing a "family of systems", each member corresponding to some Δ, or as an "uncertain system" with uncertainty Δ; we will use these terms interchangeably. The perturbation Δ is in general nonlinear, and is assumed to be passive, that is, the following holds[5]:

[5]For precise technical definitions, mathematical preliminaries, and the notation used here, see Refs. 25 and 37.

For some $\epsilon \geq 0$ and $\beta \in \mathbf{R}$,

$$\int_0^T u(t)^T (\Delta u)(t)\, dt \geq \epsilon \int_0^T u(t)^T u(t)\, dt - \beta \text{ for all } T \geq 0 \text{ and all } u \in \mathbf{L}_2.$$

We let $\mathcal{D}$ denote the set of all passive operators Δ.

Remark 4.1 Another frequently encountered class of uncertainties are *bounded* uncertainties, that is, those Δ with an $\mathbf{L}_2$ gain that does not exceed some $\gamma > 0$. However, it is well-known (for example, see Ref. 38) that under standard assumptions, one may use loop transformations to transform the case of norm-bounded Δ to the case of passive Δ considered here.

We say that the family of systems (7) is robustly stable if with w identically zero, for every member of the family and for every initial condition, every solution $x(t)$ of (7) satisfies $\lim_{t\to\infty} x(t) = 0$. Our first objective is to show how the robust stability analysis of the family of systems (7) can be performed using LMI optimization.

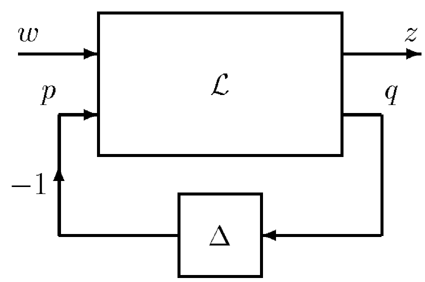
4.1 Robust Stability from the Passivity Theorem

Since Δ is known to be passive, the passivity theorem (Ref. 25) can be invoked to establish robust stability of the system (7):

> The uncertain system (7) is robustly stable if the (linear) map from p to q is strictly passive, that is, there exist $\epsilon > 0$ and $\beta \in \mathbf{R}$ such that for all $p \in \mathbf{L}_2$, the condition
>
> $$\int_0^T p(t)^T q(t)\, dt \geq \epsilon \int_0^T p(t)^T p(t)\, dt - \beta \text{ for all } T \geq 0$$
>
> holds, where p and q satisfy (7a) with w identically zero.

Strict passivity of the map from p to q is equivalent (Ref. 25) to the frequency-domain condition that for some $\epsilon > 0$,

$$G(j\omega) + G(j\omega)^* \geq 2\epsilon I \quad \text{for all } \omega \in \mathbf{R}. \tag{8}$$


w
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(If this holds, we will say with some abuse of terminology that "$G(s)$ is strictly passive".) This condition can be numerically verified (approximately) by checking that the minimum eigenvalue of $G(j\omega) + G(j\omega)^*$, sampled at a number of frequencies, is positive. An alternate condition for strict passivity that avoids frequency sampling is expressed in terms of an LMI.

Lemma 4.1 Let $(A,\, B_p,\, C_q,\, D_{qp})$ be a state-space realization of a stable linear system with transfer function $G(s)$. Then, G satisfies (8) if and only if there exists $P = P^T > 0$ satisfying the LMI

$$\begin{bmatrix} A^TP + PA & PB_p - C_q^T \\ B_p^TP - C_q & -(D_{qp} + D_{qp}^T) \end{bmatrix} < 0. \tag{9}$$

This is simply the Kalman-Yakubovich-Popov Lemma or the Positive-Real Lemma (see for example, Ref. 39, Ref. 37, pages 474–478, and Ref. 40, Chapters 5–7; see also Ref. 41).

Remark 4.2 We also have the following extension of Lemma 4.1. Let $(A,\, B_p,\, C_q,\, D_{qp})$ be a realization of a not necessarily stable linear system with transfer function $G(s)$, with A having no eigenvalues on the imaginary axis. Then, G satisfies condition (8) if and only if there exists $P = P^T$ satisfying the LMI

$$\begin{bmatrix} A^TP + PA & PB_p - C_q^T \\ B_p^TP - C_q & -(D_{qp} + D_{qp}^T) \end{bmatrix} < 0. \tag{10}$$

4.2 Multiplier Analysis

Suppose $\Delta \in \mathcal{D}$ also possesses additional properties, such as being diagonal or block-diagonal, sector-bounded memoryless, linear time-invariant or parametric, etc. Let $\mathcal{D}_{\mathrm{struct}} \subseteq \mathcal{D}$ denote the set of Δ satisfying these additional properties. (We will see two specific cases for $\mathcal{D}_{\mathrm{struct}}$ in Example 5.1 and Example 5.2.) Then the passivity theorem yields only a sufficient condition for robust stability. In this case, we can employ *multiplier theory* to utilize the additional information on Δ in order to obtain less conservative conditions for robust stability; see for example, Refs. 25 and 31–33.

Consider the system in Fig. 2, where $W_+(s)$ and $W_-(s)$ are transfer functions of some finite-dimensional LTI systems. Moreover, suppose that $W_+(s)$ and $W_-(s)$ satisfy

$$W_+(s) \text{ and } W_-(-s) \text{ are stable with stable inverses.} \tag{11}$$

The equations governing the system in Fig. 2 are:

$$\frac{d}{dt}x(t) = Ax(t) + B_p p(t) + B_w w(t), \quad q(t) = C_q x(t) + D_{qp} p(t), \quad z(t) = C_z x(t), \tag{12a}$$

$$p(t) = -\Delta(q,t), \quad \hat{p}(t) = (w_-^{\sim} \star p)(t), \quad \hat{q}(t) = (w_+ \star q)(t), \tag{12b}$$

where $w_-^{\sim}$ and w_+ are the inverse Laplace transforms of $W_-(-s)^T$ and $W_+(s)$ respectively, and "$\star$" denotes convolution.

We have the obvious but important connection between the solutions of the equations (12) and (7).

Lemma 4.2 For every $\Delta \in \mathcal{D}_{\mathrm{struct}}$, x satisfies (7) if and only if it satisfies (12) for some (every) W_+ and W_- satisfying condition (11). Therefore, system (7) is robustly stable if and only if system (12) is robustly stable, for some (every) W_+ and W_- satisfying condition (11).

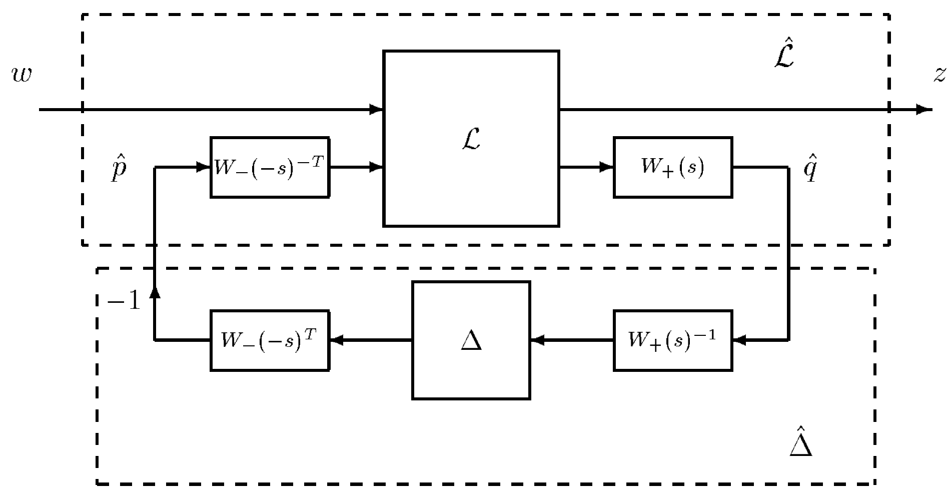
Next, suppose that

$$\text{for every } \Delta \in \mathcal{D}_{\mathrm{struct}},\ W_-(-s)^T \circ \Delta \circ W_+(s)^{-1} \text{ is passive.} \tag{13}$$

Then from the passivity theorem, system (12) is stable if

$$W_+(s)G(s)W_-(-s)^{-T} \text{ is strictly passive.} \tag{14}$$

Thus, if we find *some* W_+ and W_- such that conditions (13) and (14) hold, we have then established the robust stability of system (7). Robust stability methods using multiplier theory involve systematically searching for W_+ and W_- satisfying (11) such that conditions (13)



w
$\hat{\mathcal{L}}$
z
$\mathcal{L}$
$\hat{p}$
$W_-(-s)^{-T}$
$W_+(s)$
$\hat{q}$
-1
$W_-(-s)^T$
Δ
$W_+(s)^{-1}$
$\hat{\Delta}$

and (14) hold. Every pair of such W_+ and W_- is called a "multiplier pair proving robust stability".

It turns out that searching directly for W_+ and W_- is numerically "hard", as the set of multiplier pairs that prove robust stability is nonconvex in general.

Remark 4.3 Numerical counterexamples that establish that the set of constant multiplier pairs that prove robust stability is not necessarily convex are easy to find. Here is a simple one, with G being a linear system with no dynamics, i.e., with transfer matrix a constant:

$$G(s) = \begin{bmatrix} 1.30 & 0.30 & -0.40 \\ 0.20 & 0.50 & 0.10 \\ -0.10 & -0.60 & 0.80 \end{bmatrix}.$$

For this system, it is easily checked numerically that the multiplier factors with no dynamics

$$\left(\begin{bmatrix} 0.70 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 1.3 \end{bmatrix}, \begin{bmatrix} 0.70 & 0 & 0 \\ 0 & 1.0 & 0 \\ 0 & 0 & 1.3 \end{bmatrix} \right) \quad \text{and} \quad \left(\begin{bmatrix} 1.2 & 0 & 0 \\ 0 & -1.5 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & -1.5 & 0 \\ 0 & 0 & 0.3 \end{bmatrix} \right)$$

both prove robust stability, but their average

$$\left(\begin{bmatrix} 0.95 & 0 & 0 \\ 0 & -0.25 & 0 \\ 0 & 0 & 0.80 \end{bmatrix}, \begin{bmatrix} 0.95 & 0 & 0 \\ 0 & -0.25 & 0 \\ 0 & 0 & 0.80 \end{bmatrix}\right)$$

does not.

4.3 Robust Stability via LMI Feasibility Tests

A simple change of variables enables the reformulation of the problem of search of multiplier pairs into a convex feasibility problem. The central idea underlying the reformulation is that we should regard the product $W(s) \stackrel{\Delta}{=} W_-(s)W_+(s)$ as the variable; this product, which we shall denote by $W(s)$ is called the stability multiplier or just "multiplier". We will say that the multiplier W "proves robust stability" if it can be factored into $W(s) = W_-(s)W_+(s)$, with the pair $\{W_-, W_+\}$ proving robust stability, i.e., such that conditions (11), (13) and (14) hold. It can be shown (see Refs. 33 and 42 for details) that these conditions are equivalent, in several important cases, to the following conditions:

- W should have a certain *structure* (e.g., diagonal or block-diagonal) and *nature* (e.g., constant, Hermitian on the imaginary axis, etc.), depending on the additional information known about $\mathcal{D}_{\text{struct}}$. (For specific examples, see Ref. 33.) (15)
- For some $\epsilon > 0$, $W(j\omega) + W(j\omega)^* \geq 2\epsilon I$ for all $\omega \in \mathbf{R}$. (16)
- For some $\epsilon > 0$, $W(j\omega)G(j\omega) + G(j\omega)^*W(j\omega)^* \geq 2\epsilon I$ for all $\omega \in \mathbf{R}$. (17)

Thus, establishing robust stability using multiplier theory involves numerically searching for a multiplier W that satisfies conditions (15)–(17) above. The important observation regarding this reformulation is the following.

> *Finding W that satisfies conditions (15)–(17) is a (possibly infinite-dimensional) convex feasibility problem.*

Additionally restricting W, if necessary, to lie in a finite-dimensional subspace results in a *finite-dimensional* convex feasibility problem. Specifically, suppose that W is restricted to lie in the subspace

$$\mathcal{W} \triangleq \left\{ \sum_{i=1}^{m} \theta_i W_i \;\middle|\; \theta \in \mathbf{R}^m \right\}, \tag{18}$$

where W_i are fixed transfer matrices, each satisfying the structure and nature condition that is to be satisfied by W (see (15)). Then, every $W \in \mathcal{W}$ has a realization (A_{W}, B_{W}, $C_{\mathrm{W}}(\theta)$, $D_{\mathrm{W}}(\theta)$), with C_{W} and D_{W} being *linear* functions of θ. Using the remark following Lemma 4.1, we conclude that condition (16) is equivalent to the LMI in $P_{\mathrm{W}} = P_{\mathrm{W}}^T$ and θ:

$$\begin{bmatrix} A_{\mathrm{W}}^T P_{\mathrm{W}} + P_{\mathrm{W}} A_{\mathrm{W}} & P_{\mathrm{W}} B_{\mathrm{W}} - C_{\mathrm{W}}(\theta)^T \\ B_{\mathrm{W}}^T P_{\mathrm{W}} - C_{\mathrm{W}}(\theta) & -(D_{\mathrm{W}}(\theta) + D_{\mathrm{W}}(\theta)^T) \end{bmatrix} < 0. \tag{19}$$

Next, WG has a state-space realization (A_{WG}, B_{WG}, C_{WG}, D_{WG}) where

$$A_{WG} = \begin{bmatrix} A & 0 \\ B_{\mathrm{W}} C & A_{\mathrm{W}} \end{bmatrix}, \quad B_{WG}(\theta) = \begin{bmatrix} B \\ B_{\mathrm{W}} D \end{bmatrix},$$
$$C_{WG} = [D_{\mathrm{W}}(\theta) C \quad C_{\mathrm{W}}(\theta)], \quad D_{WG}(\theta) = D_{\mathrm{W}}(\theta) D.$$

Note that C_{WG} and D_{WG} are linear functions of θ. Then, checking condition (17) is equivalent to the LMI in $P = P^T$ and θ:

$$\begin{bmatrix} A_{WG}^T P + P A_{WG} & P B_{WG} - C_{WG}(\theta)^T \\ B_{WG}^T P - C_{WG}(\theta) & -\left(D_{WG}(\theta) + D_{WG}(\theta)^T\right) \end{bmatrix} < 0. \tag{20}$$

Thus sufficient conditions for robust stability, in several important cases, can be posed as the following feasibility problem:

$$\textit{Find } P_{\mathrm{W}},\ P \textit{ and } \theta \textit{ such that the LMIs (19) and (20) hold.} \tag{21}$$

The above formulation of robust stability tests is important in several different ways:

(i) Robust stability can be ascertained using state-space techniques and reliable convex optimization; there is no frequency sampling involved with LMI tests.

(ii) A number of classical and modern robust stability tests can be unified in this setting (for example, several tests for absolute stability, as well as the modern μ tests; see Ref. 33).

(iii) New tests can be devised, when other assumptions on the uncertainties are in effect (see for example, Ref. 43, as well as Ref. 35).

In addition, the LMI tests for robust stability can be shown to yield Lyapunov functions that "prove" robust stability (see for example Ref. 36). These Lyapunov functions can in turn be used to derive bounds on robust performance. We describe this next.

5 Robust Performance Bounds from Lyapunov Functions

The multiplier-based robust stability condition (21) can be shown to yield a positive-definite function $V : \mathbf{R}^n \times \mathbf{R}_+ \to \mathbf{R}_+$ such that $dV(x,t)/dt < 0$ along the trajectories of (7). (This function V can be constructed from the LMI variables P, P_{W} and θ; we will see a demonstration of this in Example 5.1 and Example 5.2.) Using standard arguments from Lyapunov theory, it can be shown that V is a Lyapunov function that provides a *certificate* or *guarantee* of robust stability. Additional conditions on the Lyapunov function can be imposed, yielding robust performance bounds as follows (see for example Ref. 44, Section 6, and Ref. 5): For a given initial condition $x(0)$ for the linear part (7a), suppose that the Lyapunov function V satisfies

$$\frac{d}{dt}V(x,t) \leq -\phi_\alpha(x(t), w(t), z(t)) \text{ along the trajectories of (7).} \tag{22}$$

Then it is a simple exercise to show that

$$V(x(0),0) \geq \int_0^\infty \phi_\alpha(x(t), w(t), z(t))\, dt, \tag{23}$$

or we have an upper bound on the performance index

$$\mathcal{P}_\alpha \triangleq \int_0^\infty \phi_\alpha(x(t), w(t), z(t))\, dt. \tag{24}$$

Thus, if we can find a Lyapunov function V satisfying (22), then we can obtain upper bounds on robust performance indices of the form (24), for system (7). (A variation of this technique can be used to obtain bounds on more general performance measures; see for example Ref. 5, Chapters 5–6.)

We will see in Example 5.1 and Example 5.2 that imposing condition (22) on the Lyapunov functions obtained from multiplier theory leads to LMI conditions in several important cases. Consequently, in all these cases, the problem of computing the optimal (or smallest) robust performance bound can be reformulated as an LMI optimization problem.

Example 5.1 Unstructured Passive Δ, Bound on the Output Energy

Consider the case when Δ is a general passive operator, with the performance analysis

problem being:

With exogenous input $w = 0$, find a bound on the energy of the output, i.e., $\int_0^\infty z(t)^T z(t)\,dt$, for a given initial condition $x(0)$ of the state of the linear part of the system.

In order to derive upper bounds on $\int_0^\infty z(t)^T z(t)\,dt$, we seek Lyapunov functions that satisfy condition (22) with $\phi_\alpha(x, w, z) = z^T z$, i.e.,

$$\frac{d}{dt}V(x,t) \le -z(t)^T z(t). \tag{25}$$

Then $V(x(0), 0)$ yields an upper bound on the output energy.

Since Δ is a general passive operator, the only possible multiplier pair is given by $W_+ = W_- = I$. The corresponding Lyapunov functions turn out (see Refs. 45 and 46) to be of the form

$$V(x,t) = x(t)^T P x(t) - 2\int_0^t p(\tau)^T q(\tau)\,d\tau,$$

where $P > 0$. Condition (25) holds along the trajectories of system (7) if the following condition holds:

$$\begin{bmatrix} A^TP + PA + C_z^TC_z & PB_p - C_q^T \\ B_p^TP - C_q & -(D_{qp} + D_{qp}^T) \end{bmatrix} \leq 0. \qquad (26)$$

Thus, the best upper bound on the energy of z, for a given initial condition $x(0)$ of the linear part of the system, is simply

$$\min \quad x(0)^TPx(0)$$

$$\text{s.t.} \quad P > 0, \text{ and } (26)$$

This is an "eigenvalue problem" (see Section 2.1).

Example 5.2 Diagonal, Time-Invariant, Sector-Bounded, Memoryless Nonlinearities Δ, Bound on the State Variables

Next, consider the case when: D_{qp} is zero; the uncertainty Δ is diagonal, i.e., $p_i(t) = -\delta_i(q_i(t))$; and each δ_i is a time-invariant, memoryless nonlinearity in sector $[0, \infty)$ (see Ref. 25 for the definitions of the various terms used here). For this case, the performance analysis problem considered is:

With exogenous input $w = 0$, find bounds on the state $x(t)$, $t \geq 0$, of the linear part of the system, for a given initial condition $x(0)$.

In other words, we seek an invariant set for the trajectories $x(t)$ of the linear part of the system. For every positive-definite function V for which $dV(x,t)/dt \leq 0$ holds along the trajectories of system (7), we have

$$V(x(t),t) \leq V(x(0),0) \text{ for } t \geq 0.$$

This inequality can then be used to derive bounds on $x(t)$.

For the case of diagonal, time-invariant, sector-bounded, memoryless nonlinearities, the multiplier pairs for robust stability turn out to be $W_- = I$ and

$$W_+(s) = \mathbf{diag}\left((\lambda_1 + \mu_1 s), \ldots, (\lambda_m + \mu_m s)\right),$$

where $\mu_i \geq 0$, $\lambda_i > 0$. (This corresponds to a multivariable version of the Popov criterion (Ref. 25).) The corresponding Lyapunov functions are of the form

$$V(x,t) = x(t)^T P x(t) + 2\sum_{i=1}^{m} \mu_i \int_0^{q_i(t)} \delta_i(\sigma)\, d\sigma - 2\sum_{i=1}^{m} \lambda_i \int_0^t p_i(\tau) q_i(\tau)\, d\tau,$$

where $P > 0$, $\mu_i \geq 0$ and $\lambda_i > 0$. In this case, condition $dV(x,t)/dt \leq 0$ is equivalent to the

LMI

$$\begin{bmatrix} A^TP + PA & PB_p - (\Lambda C_q + MC_qA)^T \\ B_p^TP - (\Lambda C_q + MC_qA) & -(MC_qB_p + B_p^TC_q^TM) \end{bmatrix} \leq 0 \qquad (27)$$

where $P > 0$, $M = \mathbf{diag}\,(\mu_1, \ldots, \mu_m) \geq 0$ and $\Lambda = \mathbf{diag}\,(\lambda_1, \ldots, \lambda_m) > 0$. If condition (27) holds, then the ellipsoid

$$\mathcal{E} = \left\{ \psi \in \mathbf{R}^n \;\middle|\; \psi^TP\psi \leq x(0)^TPx(0) \right\}$$

is an invariant ellipsoid for the state x.

The problem of finding the smallest invariant ellipsoid (using our techniques), i.e., one with the smallest major axis, can be solved by solving the "eigenvalue problem" (see Section 2.1)

$$\begin{array}{ll} \max & \nu \\ \text{s.t.} & P > \nu I,\;\; x(0)^TPx(0) \leq 1,\;\; (27), \\ & M = \mathbf{diag}\,(\mu_1, \ldots, \mu_m) \geq 0,\;\; \Lambda = \mathbf{diag}\,(\lambda_1, \ldots, \lambda_m) > 0 \end{array}$$

With P_{opt} denoting the optimal P, the smallest invariant ellipsoid that contains the state $x(t)$ for all $t \geq 0$ is

$$\left\{ \psi \in \mathbf{R}^n \;\middle|\; \psi^TP_{\text{opt}}\psi \leq 1 \right\}.$$

6 Conclusions

We have provided an introduction to optimization based on Linear Matrix Inequalities, and demonstrated its application on one important problem in robust control, that of robust stability and performance analysis of uncertain systems. The popularity of LMI methods, and convex optimization in general, continues to grow with new applications being discovered in other engineering disciplines besides control (for example, truss topology design (Ref. 47) and VLSI design (Refs. 48 and 49)). This popularity can be directly traced to the ease with which convex optimization problems can be numerically solved; indeed it can be argued that the reduction of an engineering problem to a convex optimization problem constitutes a "solution". The scope of the application of LMI optimization to engineering problems will only grow with further increases in computing power and advances in optimization theory.

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