SYNTHESIS OF FIXED-STRUCTURE CONTROLLERS VIA NUMERICAL OPTIMIZATION

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ABSTRACT

We propose an iterative algorithm for designing linear time-invariant controllers with some prespecified structure. The iterations require the solution of optimization problems based on Linear Matrix Inequalities, in which either the Lyapunov function proving a certain property or the controller to be designed is alternately regarded as the optimization variable (while the other is fixed). A number of structure constraints on the controller (reduced-order, decentralized, etc) can be addressed using this technique, which also extends to plants with nonlinearities or uncertainties.

The algorithm is heuristic in nature, and is not guaranteed to converge globally. However it provides a locally optimal solution which depends on the initialization of the algorithm, and serves as a useful design tool.

1. Introduction

The problem of synthesizing optimal linear full-order controllers for linear time-invariant (LTI) systems is a well-studied problem. With various criteria of optimality, a number of analytical solutions have been derived for the full-order optimal controller design problem, most of them based on Riccati equations (see for example, [5]), or polynomial factorizations (see for example, [9]). However, constraints such as fixed or reduced controller order, decentralized structure, etc, cannot be handled by these approaches. Indeed, there appears to be widespread belief that design of optimal fixed-structure controllers is a *much harder* problem[1] than that of design of full-order controllers. The same is true for the multimodel control problem, in which a single controller that operates satisfactorily with a number of different plants is sought.

Several heuristic procedures have been proposed for these more difficult problems; most of them rely on numerical optimization [11, 12]. In this paper, we present one such method — a systematic procedure for solving a number of controller design problems when structure constraints are imposed. The approach is based on a two-stage optimization process reminiscent of the D–K iteration procedure used in μ-synthesis [10] and the approach in [17]. Each stage requires the solution of a convex optimization problem, more specifically one based on Linear Matrix Inequalities (LMIs),[2] in which either the controller gain matrix, or the Lyapunov function that proves a certain property is considered the optimization variable (while the other is fixed).

The outline of the paper is as follows. In §2,

[1]Some of these problems can be reduced to Linear Matrix Inequality problems with additional rank constraints, which are known to be NP-hard.

[2]For the definition of a Linear Matrix Inequality, as well as a description of some optimization problems based on LMIs, see [4].

we present the fixed order controller design problem, and in §3 we describe the idea underlying the design algorithm presented in the paper, along with a few common design objectives. The algorithm we present is heuristic, and is not guaranteed to converge globally. Therefore, we will not discuss any convergence issues, except to state that the algorithm is a local optimization procedure, and has the potential to yield good designs depending on its initialization. We present a simple example in §4, some extensions in §5, and conclude with §6.

2. Problem Description

We consider the linear time-invariant (LTI) system described by

$$\begin{aligned} \dot{x} &= Ax + B_w w + B_u u, \\ z &= C_z x + D_{zw} w + D_{zu} u, \\ y &= C_y x, \end{aligned}$$

where $x : \mathbf{R}_+ \to \mathbf{R}^n$ is the state, $w : \mathbf{R}_+ \to \mathbf{R}^{n_w}$ the exogenous input, $z : \mathbf{R}_+ \to \mathbf{R}^{n_z}$ the output of interest, $u : \mathbf{R}_+ \to \mathbf{R}^{n_u}$ the control input, and $y : \mathbf{R}_+ \to \mathbf{R}^{n_y}$ the measured output.

For this system, we consider LTI controllers

$$\begin{aligned} \dot{\bar{x}} &= \bar{A}\bar{x} + \bar{B}_y y, \qquad \bar{x}(0) = 0, \\ u &= \bar{C}_u \bar{x} + \bar{D}_{uy} y, \end{aligned}$$

where $\bar{x} : \mathbf{R}_+ \to \mathbf{R}^n_c$, and where the controller matrix

$$K = \begin{bmatrix} \bar{A} & \bar{B}_y \\ \bar{C}_u & \bar{D}_{uy} \end{bmatrix} \tag{1}$$

has some prescribed structure. The controller design problem is determining K such that the closed-loop system described by the following equations has certain desirable properties.

$$\begin{aligned} \dot{\tilde{x}} &= (\tilde{A} + \tilde{B}_u K \tilde{C}_y)\tilde{x} + \tilde{B}_w w, \\ z &= (\tilde{C}_z + \tilde{D}_{zu} K \tilde{C}_y)\tilde{x} + D_{zw} w, \end{aligned}$$

where

$$\tilde{x} = \begin{bmatrix} x \\ \bar{x} \end{bmatrix}, \; \tilde{A} = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix},$$

$$\tilde{B}_u = \begin{bmatrix} 0 & B_u \\ I & 0 \end{bmatrix}, \; \tilde{B}_w = \begin{bmatrix} B_w \\ 0 \end{bmatrix},$$

$$\tilde{C}_y = \begin{bmatrix} 0 & I \\ C_y & 0 \end{bmatrix}, \; \tilde{D}_{zu} = \begin{bmatrix} 0 & D_{zu} \end{bmatrix}.$$

3. Controller Design Problems

We now consider a number of common design objectives.

3.1 Stability

Let us first consider the requirement that the controller yields a stable closed-loop system (2), that is, there exists a matrix $\tilde{P} \in \mathbf{R}^{(n+n_c)\times(n+n_c)}$ such that

$$\begin{gathered} \tilde{P} > 0, \\ (\tilde{A} + \tilde{B}_u K \tilde{C}_y)^T \tilde{P} + \tilde{P}(\tilde{A} + \tilde{B}_u K \tilde{C}_y) < 0. \end{gathered} \tag{2}$$

Obviously, this is a condition that a matrix inequality (in variables $\tilde{P}$ and K) be feasible. If K is a *full* matrix given by (1), and if the number of states n_c in the controller equals that of the plant n, then it can be shown that condition (2) can be written as a (convex) Linear Matrix Inequality that does not involve K: Let $\tilde{Q} = \tilde{P}^{-1}$, and denote by P (resp. Q) the upper left $n \times n$ block of $\tilde{P}$ (resp. $\tilde{Q}$). It can be shown that a necessary and sufficient condition of the existence of a stabilizing controller is that there exist P, Q and a scalar σ such that

$$\begin{bmatrix} P & I \\ I & Q \end{bmatrix} \geq 0$$

and

$$\begin{aligned} A^T P + PA &< \sigma C_y^T C_y, \\ AQ + QA^T &< \sigma B_u B_u^T. \end{aligned}$$

This fact is significant since optimization problems involving LMIs (such as feasibility of an LMI) are "easy" to solve; see [4] for specific details.

However, if K is not any full matrix, but has some structure constraint imposed on it — the structure of K being determined by the requirements on the structure of the controller — then the matrix inequality is not convex[3] in K and P. The same holds when the controller is required to be of an order strictly smaller than that of the plant.

No globally convergent, polynomial-time methods are known for finding feasible points for these nonconvex matrix inequalities. We therefore propose a heuristic approach of alternately solving convex optimization problems, more specifically those based on LMIs. The approach is this: We minimize σ, over K and $\tilde{P}$, subject to the constraints

$$\begin{gathered} \tilde{P} > 0, \\ (\tilde{A} + \tilde{B}_u K \tilde{C}_y)^T \tilde{P} + \tilde{P}(\tilde{A} + \tilde{B}_u K \tilde{C}_y) < \sigma I. \end{gathered} \tag{3}$$

Clearly, the closed-loop system is stable if and only if the minimum value of σ is negative.

This problem is a convex optimization problem in $\tilde{P}$ and σ for fixed K, and is convex in K and σ for fixed $\tilde{P}$. This suggests that we may alternate between solving convex LMI optimization problems in order to solve the matrix inequality (2). This is what is termed as a "V–K" iteration (this is akin to the D–K iteration of μ-synthesis), since we alternately solve convex optimization problems, by either fixing the Lyapunov function V (in this case the positive definite matrix $\tilde{P}$ characterizing the quadratic Lyapunov function $V(\psi) = \psi^T \tilde{P} \psi$) or the controller K.

The algorithm proceeds as follows.

```
k_iter = 0;  P = I;  K = 0;
repeat {
  if k_iter is even, solve LMI
  problem, "minimize σ, over P and
  σ, subject to (3)";  P = P_opt;

  if k_iter is odd, solve LMI problem,
  "minimize σ, over K and σ,
  subject to (3)";  K = K_opt;

  k_iter = k_iter + 1;
  if σ < 0, exit;
}
```

[3]Inequality (2) is a so-called Bilinear Matrix Inequality (see [13] and the references therein for details).

We reiterate that this approach is a heuristic way of solving a nonconvex optimization problem. It is guaranteed to converge, but not necessarily to the global optimum of the problem. In particular, if a step of the V–K iteration is found infeasible, then it is possible that a feasible controller still exists. Each step of the iteration can be very efficiently performed via recently developed algorithms for convex optimization over LMIs [16, 3, 15]. For more on problems in systems and control that can be solved via optimization over LMIs, we refer the reader to [4] and the references therein.

3.2 Decay rate

Suppose we wish to maximize the decay rate of system (2) over output feedback matrices K, subject to a Euclidean norm constraint[4] on K of the form $\mathrm{Tr}\, K^T K < \alpha$. Even when K is a full matrix, this problem has no analytical solution, nor can it be posed as a convex optimization problem. However, we can solve this problem using V–K iterations.

The decay rate of the system (2) exceeds λ if and only there exists $P > 0$ such that

$$\begin{aligned} &\mathrm{Tr}\, K^T K < \alpha \\ &(\tilde{A} + \tilde{B}_u K \tilde{C}_y)^T P + P(\tilde{A} + \tilde{B}_u K \tilde{C}_y) \\ &\qquad < \lambda P. \end{aligned}$$

We note that for fixed P, the inequalities can be rewritten as LMIs in K and λ, and for fixed K and λ, they are LMIs in P. We can therefore use a V–K iteration scheme similar to the one outlined in the previous subsection to design K so as to (locally) maximize the decay rate.

[4]Without this constraint, the decay rate maximization problem typically becomes ill-posed, in the sense that the decay rate can be made as large as needed by letting K be correspondingly "large".

3.3 $\mathbf{H}_2$ norm

The $\mathbf{H}_2$ norm from w to z for system (2) is less than η if and only if $D_{zw} = 0$ and there exists $\tilde{Q} > 0$ such that

$$(\tilde{A} + \tilde{B}_u K \tilde{C}_y)\tilde{Q} + \tilde{Q}(\tilde{A} + \tilde{B}_u K \tilde{C}_y)^T + \tilde{B}_w \tilde{B}_w^T \leq 0,$$

$$\mathbf{Tr}\ (\tilde{C}_z + \tilde{D}_{zu} K \tilde{C}_y)\tilde{Q}(\tilde{C}_z + \tilde{D}_{zu} K \tilde{C}_y)^T < \eta^2.$$

The second constraint can be reformulated as a matrix inequality which is linear in K and η^2 for fixed $\tilde{Q}$ and linear in $\tilde{Q}$ and η^2 for fixed K. Once again, we can minimize η over K and $\tilde{Q}$ using V–K iterations.

3.4 $\mathbf{H}_\infty$ norm

When $D_{zw} = 0$, the $\mathbf{H}_\infty$ norm from w to z for system (2) is less than γ if and only there exists $\tilde{P} > 0$ such that

$$(\tilde{A} + \tilde{B}_u K \tilde{C}_y)^T \tilde{P} + \tilde{P}(\tilde{A} + \tilde{B}_u K \tilde{C}_y) + (\tilde{C}_z + \tilde{D}_{zu} K \tilde{C}_y)^T (\tilde{C}_z + \tilde{D}_{zu} K \tilde{C}_y) + \frac{1}{\gamma^2} \tilde{P} \tilde{B}_w \tilde{B}_w^T \tilde{P} < 0.$$

The above constraint can be reformulated as a matrix inequality which is linear in K and γ^2 for fixed $\tilde{P}$, and linear in $\tilde{P}$ and γ^2 for fixed K. We may therefore minimize γ using V–K iterations.

4. Example

To illustrate the approach, we consider an helicopter model, as given in [1]. The LTI model is (we assume that there are no parametric disturbances)

$$\dot{x} = Ax + B_u u, \quad y = C_y x,$$

where

$$A = \begin{bmatrix} -.0366 & .0271 & .0188 & -.4555 \\ .0482 & -1.01 & .0024 & -4.0208 \\ .1002 & .3681 & -.707 & 1.42 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$B_u = \begin{bmatrix} .4422 & .1761 \\ 3.5446 & -7.5922 \\ -5.52 & 4.49 \\ 0 & 0 \end{bmatrix}, \quad C_y^T = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

We seek a static output-feedback control law $u = Ky$, where K is a 2×1 matrix that stabilizes the system.

Starting with $K = 0$, we maximize the decay rate of the system alternatively over P (characterizing the Lyapunov function $V(x) = x^T P x$) and K. We also impose a bound on the Euclidean norm of the controller matrix by requiring $K^T K \leq 20$ (this can be written as an LMI).

This is equivalent to minimizing λ subject to

$$\lambda P > (A + B_u K C_y)^T P + P(A + B_u K C_y),$$
$$P > 0, \quad \mathbf{Tr} P = 4, \quad K^T K \leq 20.$$

(The trace constraint is introduced to take into account the fact that the LMIs are homogeneous in P.) The resulting decay rate is $-\lambda/2$.

After 3 iterations in each variable P and K, we find

$$P = \begin{bmatrix} 0.2795 & -0.0837 & -0.1198 & -0.0161 \\ -0.0837 & 0.2339 & 0.3909 & 0.5378 \\ -0.1198 & 0.3909 & 1.7431 & 0.7792 \\ -0.0161 & 0.5378 & 0.7792 & 1.7434 \end{bmatrix},$$

and

$$K^T = \begin{bmatrix} 0.3385 & 2.3951 \end{bmatrix}.$$

The corresponding closed-loop eigenvalues are -18.1154, -0.1636, $-0.2294 \pm 0.7296\jmath$. The decay rate is 0.1636.

5. Some Extensions

5.1 Generalization to multimodel problems

The approach generalizes to the case when a single controller is required to operate satisfactorily with different plants. For instance, the existence of a common stabilizing controller for a finite number of LTI systems can be formulated as a V–K optimization problem. It is interesting to note that Blondel and Gevers [2] have recently shown that the general problem of checking whether a common stabilizing controller exists for more than three plants is undecidable. This suggests that our heuristic approach to this problem may be well-justified.

5.2 More general Lyapunov functions

The approach can also handle control problems in which the plant is not known or modeled exactly (i.e., "uncertain" or "with perturbations"). Depending on the additional additional assumptions on the plant uncertainty, it is possible to search for Lyapunov functions that are more general than mere quadratic forms. For instance, consider a system subject to memoryless, sector-bounded nonlinearities. In this case, it is possible to search for a Lyapunov function of the Lur'e type (i.e., consisting of a quadratic form plus an integral of the nonlinearity). Similarly, if the system is subject to (constant) parametric, unknown-but-bounded uncertainties, it is possible to search for more complicated Lyapunov functionals instead of a simple quadratic forms (see [4] and references therein for more details).

5.3 Extending the V–K iteration

The basic idea behind V–K iterations can be extended as follows: Suppose we have an optimization problem $\mathcal{P}$, with $\mathcal{X}$ denoting the set of optimization variables. We write $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2 \cup \cdots \cup \mathcal{X}_k$, with the subsets $\mathcal{X}_j$ satisfying the additional property that for $j = 1, \ldots, k$, the optimization problem $\mathcal{P}_j$, obtained from $\mathcal{P}$ by treating only the elements of $\mathcal{X}_j$ as optimization variables (with the remaining optimization variables of $\mathcal{P}$ fixed at some constant value), is "easy" to solve. Then, by cycling through the subsets $\mathcal{X}_j$, we can solve a sequence of easy problems in order to get a locally optimal solution to the original problem $\mathcal{P}$. (Obviously, the V—K iteration corresponds to the case when $k = 2$.) We hasten to add that such ideas are not new; however, the recent research on the theory of LMI optimization has added another member to the set of "easy" optimization problems, which may extend the set of problems that can be satisfactorily "solved" using such heuristics.

6. Conclusions

We have presented an algorithm for designing LTI controllers with constraints on their structure. The algorithm that we have presented is heuristic; however, it often converges to good designs, and when it does, provides a certificate (via a Lyapunov function) that the controller does satisfy the specified design constraints.

A thorough study of the convergence behavior of the algorithm remains to be done. This algorithm should also be compared against other methods for controller design such as the alternating projection method due to Grigoriadis and Skelton [7, 8, 14], as well as algorithms for solving Bilinear Matrix Inequalities [13, 6].

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