

Linear Matrix Inequalities for Signal Processing An Overview¹

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Abstract

A wide variety of problems in system theory can be formulated (or reformulated) as convex optimization problems involving linear matrix inequalities (LMIs), that is, constraints requiring an affine combination of symmetric matrices to be positive semidefinite. Important examples are the analysis of and design for uncertain systems, and optimal digital filter design and realization. For a few very special cases, there are “analytical solutions” to LMI optimization problems, but in general they can be solved numerically very efficiently. Thus, the reduction of a problem from system theory to an optimization problem based on LMIs constitutes, in a sense, a “solution” to the original problem. Our objective in this paper is to focus on the application of LMI optimization to problems from signal processing.

1 Introduction

A wide variety of problems in system theory can be reduced to a handful of standard convex and quasiconvex optimization problems that involve linear matrix inequalities or LMIs, that is constraints of the form

$$F(x) \triangleq F_0 + \sum_{i=1}^m x_i F_i > 0, \quad (1)$$

where $x \in \mathbf{R}^m$ is the variable, and $F_i = F_i^T \in \mathbf{R}^{n \times n}$, $i = 0, \dots, m$, are given. Though the form of the LMI (1) appears very specialized, it turns out that it is widely encountered in system and control theory. A comprehensive list of examples can be found in [1, 2, 3]. For a few very special cases there are “analytical solutions” to LMI optimization problems, but in general they can be solved numerically very efficiently. Indeed, the recent popularity of LMI optimization for control can be directly traced to the recent breakthroughs in interior point methods for LMI optimization (see for example, [4, 5, 6, 7]). The growing popularity of LMI methods for control is also evidenced by the large number of publications in recent control conferences.

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While much of the research effort in the application of LMI optimization has been directed towards problems from control theory, many of the underlying techniques extend to problems from other areas of engineering as well, for instance, truss topology design [8] and VLSI design [9, 10]). Our objective, in this paper, is to describe the application of LMI optimization towards the solution of problems from signal processing.

2 Optimization based on Linear Matrix Inequalities

Recall the definition of a linear matrix inequality:

$$F(x) \triangleq F_0 + \sum_{i=1}^m x_i F_i > 0,$$

where $x \in \mathbf{R}^m$ is the variable, and $F_i = F_i^T \in \mathbf{R}^{n \times n}$, $i = 0, \dots, m$ are given. The set $\{x \mid F(x) > 0\}$ is convex, and need not have smooth boundary. (We have used strict inequality mostly for convenience; inequalities of the form $F(x) \geq 0$ are also readily handled.)

Multiple LMIs $F_1(x) > 0, \dots, F_n(x) > 0$ can be expressed as the single LMI

$$\text{diag}(F_1(x), \dots, F_n(x)) > 0.$$

When the matrices F_i are diagonal, the LMI $F(x) > 0$ is just a set of linear inequalities. Nonlinear (convex) inequalities are converted to LMI form using Schur complements. The basic idea is as follows: the LMI

$$\begin{bmatrix} Q(x) & S(x) \\ S(x)^T & R(x) \end{bmatrix} > 0, \quad (2)$$

where $Q(x) = Q(x)^T$, $R(x) = R(x)^T$, and $S(x)$ depend affinely on x , is equivalent to

$$R(x) > 0, \quad Q(x) - S(x)R(x)^{-1}S(x)^T > 0. \quad (3)$$

In other words, the set of nonlinear inequalities (3) can be represented as the LMI (2).

The constraint

$$\mathbf{Tr} S(x)^T P(x)^{-1} S(x) < 1, \quad P(x) > 0, \quad (4)$$

where $P(x) = P(x)^T \in \mathbf{R}^{n \times n}$ and $S(x) \in \mathbf{R}^{n \times p}$ depend affinely on x , is handled by introducing a new (slack) matrix variable $X = X^T \in \mathbf{R}^{p \times p}$, and the LMI (in x and X):

$$\mathbf{Tr} X < 1, \quad \begin{bmatrix} X & S(x)^T \\ S(x) & P(x) \end{bmatrix} > 0. \quad (5)$$

We often encounter problems in which the variables are matrices, e.g.,

$$A^T P + P A < 0,$$

where $A \in \mathbf{R}^{n \times n}$ is given and $P = P^T$ is the variable. In this case we will not write out the LMI explicitly in the form $F(x) > 0$, but instead make clear which matrices are the variables.

Two standard LMI optimization problems are of interest:

- *LMI feasibility problem.* Given an LMI $F(x) > 0$, the corresponding LMI feasibility problem is to find x^{feas} such that $F(x^{\text{feas}}) > 0$ or determine that the LMI is infeasible. (By duality, this means: find a nonzero $G \geq 0$ such that $\mathbf{Tr} G F_i = 0$ for $i = 1, \dots, m$ and $\mathbf{Tr} G F_0 \leq 0$.) This is a convex feasibility problem.

- *Semidefinite Programming problem (SDP).* An SDP requires the minimization of a linear objective subject to LMI constraints:

$$\begin{aligned} & \text{Minimize} && c^T x \\ & \text{subject to} && F(x) > 0 \end{aligned}$$

where c is a real vector, and F is a symmetric matrix that depends affinely on the optimization variable x . This is a convex optimization problem.

Both these problems can be numerically solved very efficiently, using currently available software [11, 12]. Thus, the reduction of a problem from signal processing to an LMI problem can be thought of as providing a “solution” to the original problem.

3 Filter design constraints as LMI constraints

The problem we consider is that of designing a digital filter as a linear combination of given stable basis transfer functions:

$$H(z, \theta) = \sum_{i=1}^N \theta_i H_i(z),$$

where H_i are fixed transfer functions and θ_i are the parameters to be determined. We now describe how

a number of commonly encountered design constraints on H can be reformulated as LMI constraints on θ ; also see [13].

3.1 Pointwise frequency-domain equality constraints

Consider the constraint:

$$H(z_i, \theta) = f_i, \quad i = 1, \dots, M,$$

where z_i and f_i are some specified complex numbers. This constraint immediately yields M linear equality constraints on θ .

3.2 Pointwise norm upper bounds on the transfer function

Consider the constraint:

$$\sigma_{\max}(H(z_i, \theta)) \leq g_i, \quad i = 1, \dots, M, \quad (6)$$

where $z_i \in \mathbf{C}$ and $g_i > 0$ are specified. This constraint yields M LMI constraints on θ as follows. For every i , the constraint $\sigma_{\max}(H(z_i, \theta)) \leq g_i$ is equivalent to

$$H(z_i, \theta)^* H(z_i, \theta) \leq g_i^2 I,$$

which, from (2) and (3), is equivalent to the LMI

$$\begin{bmatrix} g_i I & H(z_i, \theta)^* \\ H(z_i, \theta) & g_i I \end{bmatrix} \geq 0.$$

Note that when $H(z, \theta)$ is a scalar transfer function, (6) represents magnitude constraints on the transfer function, and therefore results in quadratic constraints on θ . In addition, when the z_i s lie on the unit circle, (6) represents upper bounds on the frequency response magnitude.

3.3 Pointwise time-domain constraints

Consider the constraint:

The response y of the filter $H(z, \theta)$ to a reference input u_{ref} satisfies

$$y_{\text{lb}}(k) \leq y(k) \leq y_{\text{ub}}(k), \quad k = 1, \dots, N,$$

where y_{lb} and y_{ub} are specified functions.

This constraint is immediately shown to yield $2N$ linear constraints on θ as follows: With \hat{y} denoting the z -transform of y , we have

$$\hat{y} = H(z, \theta) \hat{u}_{\text{ref}},$$

so that $y(k)$ is a *linear* function of θ for every k .

3.4 Upper bounds on the \mathbf{H}_2 norm

The \mathbf{H}_2 norm of the transfer function $G(z)$ of a linear system is defined as

$$\begin{aligned} \|G\|_2 &= \sqrt{\frac{1}{2\pi} \int_0^{2\pi} \mathbf{Tr} (G(e^{j\omega}) G(e^{j\omega})^*) d\omega} \\ &= \sqrt{\sum_{k=0}^{\infty} \mathbf{Tr} (g(k) g(k)^T)}, \end{aligned} \quad (7)$$

where g is the inverse z transform of G . (Note that the \mathbf{H}_2 norm of FIR filters is simply the square-root of the sum of the squares of the filter coefficients.) The \mathbf{H}_2 norm measures the RMS value of the output of the system when it is driven by white noise with unit power spectral density. Given a state-space realization (A, B, C, D) of G , the quantity $\|G\|_2$ can be calculated as follows. Starting with (7), note that

$$\begin{aligned}\|G\|_2^2 &= \mathbf{Tr} DD^T + \sum_{k=0}^{\infty} \mathbf{Tr} (CA^k BB^T (A^T)^k C^T) \\ &= \mathbf{Tr} DD^T + \mathbf{Tr} CW_c C^T,\end{aligned}$$

where W_c is the controllability Gramian of the realization (A, B, C, D) , given as the solution to the Lyapunov equation

$$AW_c A^T - W_c + BB^T = 0. \quad (8)$$

Consider an upper bound on the \mathbf{H}_2 norm of $H(z, \theta)$:

$$\|H\|_2 \leq \mu. \quad (9)$$

We now show how this constraint yields an LMI constraint on θ . First, note that since H depends affinely on θ , there exists a state-space realization $(A, B, C(\theta), D(\theta))$ for H , where A and B are constant matrices and C and D depend affinely on θ . Then, the constraint (9) is equivalent to

$$\mathbf{Tr} D(\theta)D(\theta)^T + \mathbf{Tr} C(\theta)W_c C(\theta)^T \leq \mu^2,$$

where W_c is given by (8). From (4) and (5), this constraint is equivalent to the LMI constraint

$$\mathbf{Tr} X < \mu^2, \quad \begin{bmatrix} X & C(\theta) & D(\theta) \\ C(\theta)^T & W_c^{-1} & 0 \\ D(\theta)^T & 0 & I \end{bmatrix} > 0.$$

3.5 Upper bounds on the \mathbf{H}_∞ norm

The \mathbf{H}_∞ norm of a transfer function $G(z)$ is defined as

$$\|G\|_\infty = \sup_{|z|>1} \sigma_{\max}(G(z)).$$

The \mathbf{H}_∞ norm equals the “energy-gain” of the system: It is the largest value, over all possible inputs of unit energy, of the energy of the output of the system, i.e.,

$$\|G\|_\infty^2 = \sup_{u \neq 0} \frac{\sum_{k=0}^{\infty} y(k)^T y(k)}{\sum_{k=0}^{\infty} u(k)^T u(k)},$$

where y is the output of the system corresponding to the input u . The \mathbf{H}_∞ norm also equals the “RMS-gain” of system.

Given some $\gamma > 0$, the Bounded Real Lemma (also known as a Kalman-Yakubovich-Popov (KYP) Lemma; see for example, [1] and the references therein)

enables us to write down a LMI condition for the constraint $\|H\|_\infty < \gamma$: Given a state-space realization (A, B, C, D) of G , the condition $\|G\|_\infty < \gamma$ holds if and only if there exists a symmetric matrix P such that the following LMI holds:

$$\begin{bmatrix} A^T P A - P & A^T P B & C^T \\ B^T P A & B^T P B - \gamma I & D^T \\ C & D & -\gamma I \end{bmatrix} < 0.$$

The Bounded Real lemma immediately enables the reformulation of an upper bound on the \mathbf{H}_∞ norm of $H(z, \theta)$,

$$\|H\|_\infty \leq \gamma, \quad (10)$$

as an LMI constraint on θ . Let $(A, B, C(\theta), D(\theta))$ be a state-space realization for H , where A and B are constant matrices and C and D depend affinely on θ . Then, the constraint (10) is equivalent to the LMI constraint

$$\begin{bmatrix} A^T P A - P & A^T P B & C(\theta)^T \\ B^T P A & B^T P B - \gamma I & D(\theta)^T \\ C(\theta) & D(\theta) & -\gamma I \end{bmatrix} < 0. \quad (11)$$

Note that the LMI (11) is *equivalent* to the constraint (10), without any frequency sampling.

3.6 Passivity constraints

A system with input u and output y is said to be *passive* (*strictly passive*) if for some real β and $\epsilon \geq 0$ (resp. $\epsilon > 0$),

$$\sum_{k=0}^N u(k)^T y(k) \geq 2\epsilon \sum_{k=0}^N u(k)^T u(k) - \beta$$

for all $N \geq 0$ and all u .

When the input and output are power-conjugate quantities (i.e., their product has the interpretation of power), a strictly passive system is “dissipative”, i.e., dissipates energy. It turns out that a stable linear system with transfer function H is passive (resp. strictly passive) if and only if for some $\epsilon \geq 0$ (resp. $\epsilon > 0$),

$$H(z) + H(z)^* \geq 2\epsilon I \quad \text{for all } z \in \mathbf{C} \text{ with } |z| = 1. \quad (12)$$

This latter condition is equivalent, via the Positive-real Lemma (also known as a KYP Lemma; see for example, [1] and the references therein), to the existence of $P = P^T$ such that the LMI

$$\begin{bmatrix} A^T P A - P & A^T P B - C^T \\ B^T P A - C & B^T P B + 2\epsilon I - (D + D^T) \end{bmatrix} \leq 0$$

holds. Thus the condition that the linear system with transfer function $H(z, \theta)$ is passive is equivalent to the LMI in P , $\epsilon \geq 0$ and θ ,

$$\begin{bmatrix} A^T P A - P & A^T P B - C(\theta)^T \\ B^T P A - C(\theta) & B^T P B + 2\epsilon I \\ & -(D(\theta) + D(\theta)^T) \end{bmatrix} \leq 0, \quad (13)$$

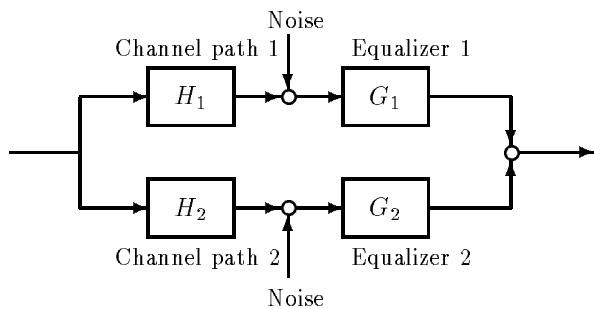


Figure 1: Schematic diagram of a two-path communication channel.

where $(A, B, C(\theta), D(\theta))$ is a state-space realization of H with A and B constant, and C and D being affine in θ . Once again, note that the LMI (13) is *equivalent* to the constraint (12), without any frequency sampling.

4 An application: Optimal Equalizer Design for Multi-Path Communication Channels

We describe an application of LMI optimization towards the design of zero-forcing equalizers for multi-path communication channels. For simplicity, we consider below a simplified version of the equalizer design problem. For more detail, we refer the reader to [14].

Consider a two-path communication channel shown in Fig. 1. Here H_1 and H_2 are estimates of the transfer functions of the two paths, realized for example through two separate antenna elements at the receiver. Sensor noises are assumed to be uncorrelated and white for simplicity.

The general design objective is to design FIR filters G_1 and G_2 so that

- G_1 and G_2 *equalize*, that is $H_1G_1 + H_2G_2 = 1$; and
- the RMS value of the output due to sensor noise is minimized.

In practice, delays inherent in the system mean that rather than having $H_1G_1 + H_2G_2 = 1$, it is more realistic to require that $H_1(z)G_1(z) + H_2(z)G_2(z) = z^{-d}$, where d is the equalizing “delay”. Moreover, in practice, the channel transfer functions H_1 and H_2 are known only approximately. Therefore, rather than equalize *exactly*, it may be advantageous to deliberately allow equalizing error, with the hope that the freedom afforded thus can be used to mitigate the effect of sensor noise. We therefore require that

$$H_1(z)G_1(z) + H_2(z)G_2(z) \approx z^{-d},$$

with the approximation error quantified by

$$\|z^{-d} - H_1(z)G_1(z) - H_2(z)G_2(z)\|_\infty.$$

(Roughly speaking, the approximation error is allowed “uniformly” across frequency.) From the discussion in §3.4, the RMS value of the output due to noise is $\|G_1\|_2^2 + \|G_2\|_2^2$, and therefore the underlying optimization problem is to design FIR G_1, G_2 of some prespecified order M , subject to

$$\|z^{-d} - H_1(z)G_1(z) - H_2(z)G_2(z)\|_\infty \leq \epsilon \quad (14)$$

$$\|G_1\|_2^2 + \|G_2\|_2^2 \leq \eta \quad (15)$$

Questions that arise are: What is the tradeoff between ϵ, η, d and M ? What are the optimal filter coefficients?

These questions are readily answered using the results in §3. Let θ be the vector consisting of the impulse response coefficients of G_1 and G_2 (i.e., the design parameters). Then, it is easy to write down a state-space realization $(A, B, C(\theta), D(\theta))$ of $z^{-d} - H_1(z)G_1(z) - H_2(z)G_2(z)$, where A and B are constant matrices and C and D depend affinely on θ . Then, from the discussion in §3.5, constraint (14) can be reformulated as an LMI with θ and ϵ as some of the optimization variables. Constraint (15) can be similarly reformulated as an LMI in variables that include θ and η , from the discussion in §3.4. Therefore, for fixed d and M , the study of the tradeoff between ϵ and η can be efficiently performed via LMI optimization. In addition, for varying d and M , one may obtain a family of tradeoff curves, yielding valuable information.

We now present a simple numerical example that illustrates these ideas. The Bode magnitude plots of the channel transfer functions are shown in Fig. 2.

We now consider the case when: (i) There is no equalization delay, i.e., $d = 0$. (ii) Each equalizer has three taps, i.e., $M = 3$. (iii) The equalization error $\epsilon = 0.001$. The design objective is to obtain G_1, G_2 so as to minimize η .

Fig. 3 shows the Bode magnitude plots of the optimal equalizers designed using LMI optimization. Fig. 4 shows the Bode magnitude plot of the resulting equalizer error transfer function. It can be seen that as expected, the peak value of the error magnitude is 0.001.

We next explore the tradeoff between ϵ and η . Fig. 5 shows the plot of the lowest achievable RMS value of the output for various values of equalization error. As expected, it can be seen that the effect of noise decreases monotonically with the allowed equalization error. While small values of permissible equalization error are sensible, allowing for very large equalization error defeats the very purpose of equalization. Indeed, it can be argued that large equalization errors correspond to allowing large intersymbol interference, which

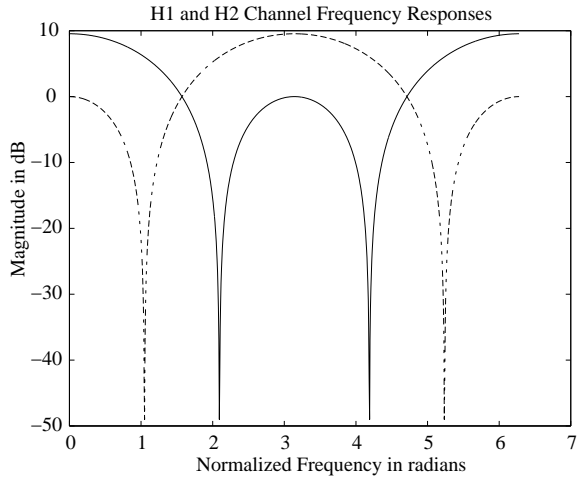


Figure 2: Bode magnitude plots of the two channel transfer functions (H_1 shown solid and H_2 shown dotted). The channels each have a notch, at $\pi/3$ and $2\pi/3$ respectively.

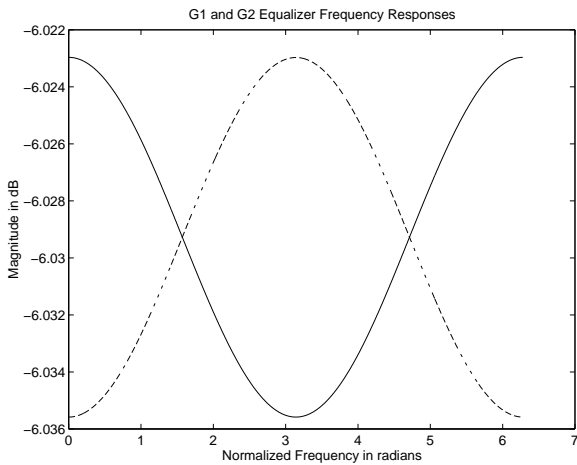


Figure 3: Bode magnitude plots of the optimal equalizer transfer functions (G_1 shown solid and G_2 shown dotted).

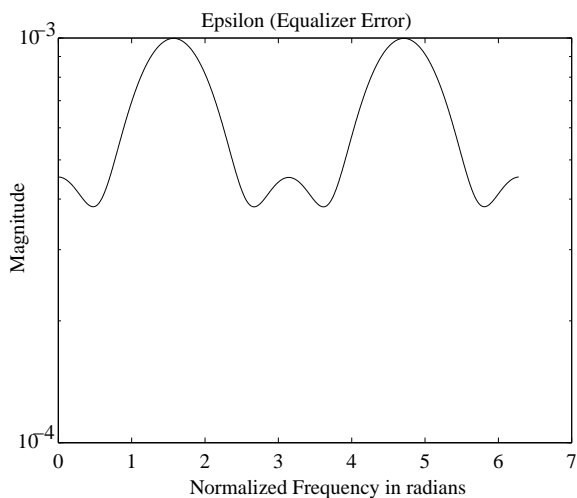


Figure 4: Bode magnitude plots of the equalizer error transfer function.

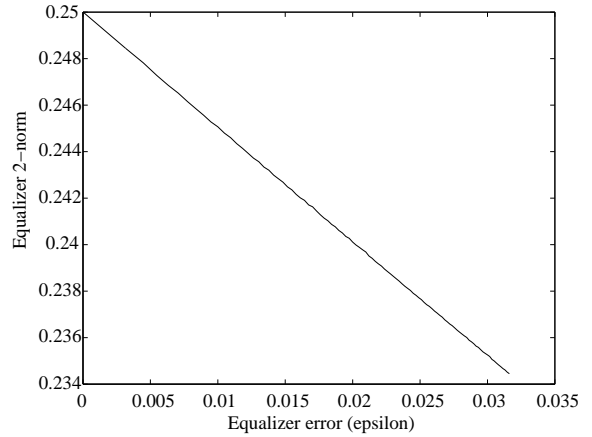


Figure 5: Bode magnitude plots of the equalizer error transfer function.

will eventually lead to poor system performance. For further discussion of this point, as well as a plot of bit error rate curves, see [14].

5 Extensions

We now present two extensions of the development in §3.

5.1 FIR filter design with upper and lower magnitude constraints

Often, it is desirable to design filters that satisfy *lower* bound constraints on the frequency response magnitude. However, the development in §3 cannot directly handle such constraints, as they turn out to be non-convex constraints on the parameters θ . As a specific example, consider the problem of designing an FIR filter with only magnitude constraints of the form

$$|\text{LB}(z)| \leq |H(z)| \leq |\text{UB}(z)|, \quad |z| = 1,$$

where $H(z) = \sum_{i=0}^{N-1} h_i z^{-i}$, and LB and UB are proper rational functions of z .

Defining $G(z) = H(z)H(1/z)$, we observe that G must satisfy

$$\text{LB}(z)\text{LB}(1/z) \leq G(z) \leq \text{UB}(z)\text{UB}(1/z), \quad |z| = 1.$$

These constraints on G can be reformulated as LMI constraints, without frequency sampling, using the Positive-Real Lemma (see §3.6); thus, G can be designed efficiently using LMI optimization. H can be recovered from G via spectral factorization [15, 16].

5.2 Minimum-phase filter design

Consider the problem of designing a stable, stably invertible filter H subject to a number of magnitude and phase constraints. Equivalently, one may design G such that $H(z) = e^{G(z)}$, with the following advantages:

- Constraints on the frequency response magnitude become constraints on $(G(z) + G(z)^*)|_{|z|=1}$.
- Constraints on the frequency response magnitude roll-off become constraints on

$$\left. \frac{d}{dz} (G(z) + G(z)^*) \right|_{|z|=1}.$$

- Constraints on the frequency response phase become constraints on $\left(\frac{G(z)}{j} + \left(\frac{G(z)}{j} \right)^* \right) \Big|_{|z|=1}$.
- Constraints on the group delay become constraints on

$$\left. \frac{d}{dz} \left(\frac{G(z)}{j} + \left(\frac{G(z)}{j} \right)^* \right) \right|_{|z|=1}.$$

All these constraints can be exactly reformulated as LMI constraints using the Positive-Real Lemma (see §3.6); once G is designed, H can be recovered as $e^{G(z)}$.

We note that while the problem of implementing $e^{G(z)}$ remains, this approach can be used to verify the feasibility of the design constraints, as well as to determine the limits of achievable designs.

6 Conclusion

Optimization based on Linear Matrix Inequalities has come to be recognized as an important tool in control. However, the very same techniques used to reformulate problems from control theory to LMI problems can be immediately applied to problems from signal processing, as we have shown in this article. Considerable research effort is currently being devoted by optimization theorists towards the numerical solution of LMI problems. This, coupled with the ever-increasing gains in computing power, means that a number of problems from signal processing can be very efficiently “solved” using LMI optimization. Issues such as the design of special-purpose hardware for solving large-scale LMI problems that arise in signal processing remain to be explored.

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