Chapter 42

Phase-sensitive structured singular value

André L. Tits* and V. Balakrishnan**

*Department of Electrical Engineering and Institute for Systems Research
University of Maryland
College Park, MD 20742
USA
andre@isr.umd.edu

**School of Electrical and Computer Engineering
Purdue University
West Lafayette, IN 47907-1285
USA
ragu@ecn.purdue.edu

42.1 Description of the problem in the simplest case

Given $\Gamma \in \mathbf{C}^{n\times n}$ with $\Gamma + \Gamma^* \geq 0$, define the phase $\Phi(\Gamma)$ of Γ by

$$\Phi(\Gamma) = \cot^{-1}\left(\sup\left\{b :\ \Gamma + \Gamma^* - \frac{\beta}{j}(\Gamma - \Gamma^*) \geq 0\ \forall\beta \in \{-b, b\}\right\}\right).$$

For $\theta \in [0, \pi/2]$, let

$$\mathbf{\Gamma}_\theta = \{\Gamma \in \mathbf{C}^{n\times n} : \Gamma + \Gamma^* \geq 0,\ \Phi(\Gamma) \leq \theta\},$$

and

$$\mathbf{BR\Delta}_\theta = \{\Delta \in \mathbf{RH}_\infty^{n\times n} :\ \|\Delta\|_\infty \leq 1,\ \Delta(\mathrm{j}\omega) \in \mathbf{\Gamma}_\theta \quad \forall \omega \in \mathbf{R}\},$$

where $\mathbf{RH}_\infty$ denotes the set of real-rational functions in $\mathbf{H}_\infty$. Following [1, 2], define the *phase-sensitive structured singular value* of $M \in \mathbf{C}^{n\times n}$ for phase θ by

$$\mu_\theta(M) = \left(\inf\{\overline{\sigma}(\Gamma) : \Gamma \in \mathbf{\Gamma}_\theta, \det(I + \Gamma M) = 0\}\right)^{-1}$$

if $\det(I + \Gamma M) = 0$ for some $\Gamma \in \mathbf{\Gamma}_\theta$, and $\mu_\theta(M) = 0$ otherwise. ($\overline{\sigma}(\Gamma)$ stands for the maximum singular value of Γ.) Also define

$$\hat{\mu}_\theta(M)$$
$$= \inf\left\{\gamma : \begin{array}{l} r(M^*M - \gamma^2 I) - ((1 + \jmath\beta)M + ((1 + \jmath\beta)M)^*) < 0, \\ \gamma > 0,\ r > 0,\ \beta \in [-\cot\theta, \cot\theta] \text{ when } \theta > 0 \end{array}\right\}. \tag{42.1}$$

For each of the following two statements, we are interested in determining conditions under which the statement holds.

1. For $P \in \mathbf{RH}_\infty^{n\times n}$, $\theta \in [0, \pi/2]$:

$$\sup_{\omega \in \mathbf{R} \cup \infty} \mu_\theta(P(\mathrm{j}\omega)) < 1 \tag{42.2}$$

   if, and only if, $(I + \Delta P)^{-1} \in \mathbf{H}_\infty$ for all $\Delta \in \mathbf{BR\Delta}_\theta$ and

$$\sup_{\Delta \in \mathbf{BR\Delta}_\theta} \|(I + \Delta P)^{-1}\|_\infty < \infty.$$

2. For $M \in \mathbf{C}^{n\times n}$, $\theta \in [0, \pi/2]$:

$$\mu_\theta(M) = \hat{\mu}_\theta(M).$$

42.2 Motivations

The phase-sensitive structured singular value was introduced in [1, 2] as a tool for the analysis of robust stability when, in addition to a magnitude bound, certain phase information is available concerning the uncertainty; see also [3]. Indeed, it can be checked that a matrix Γ belongs to $\mathbf{\Gamma}_\theta$ if, and only if, its numerical range (field of values) is contained in a sector of aperture 2θ about the positive real axis. For example, $\theta = \pi/2$ corresponds to the case when the uncertainty is known to be passive. For given M and θ, $\hat{\mu}_\theta(M)$ can be evaluated by solving a linear matrix inequality GEVP (generalized eigenvalue minimization problem, [2, 4]). It is easily shown that $\hat{\mu}_\theta(M)$ is always an upper bound to $\mu_\theta(M)$. The first of the statements above, when it holds, is a "small-μ theorem." The second statement, when it holds, implies that exact computation of $\mu_\theta(M)$ is tractable.

42.3 More general formulation

The case of block-diagonal structures, with different (possibly frequency-dependent) phase bounds on each block, is of interest as well. Assume ℓ blocks of size $k_1, \ldots, k_\ell$, let $\Theta = (\theta_1, \ldots, \theta_\ell)$ with $\theta_i \in [0, \pi/2]$ and define $\mathbf{\Gamma}_\Theta$ and $\mu_\Theta(M)$ in the obvious way. Then

$$\hat{\mu}_\Theta(M) = \inf \left\{ \gamma : \begin{array}{l} M^*RM - \gamma^2 R - S(I + \mathrm{j}B)M - M^*(I - \mathrm{j}B)S < 0, \\ \gamma > 0,\ R,\ S \in \mathcal{S},\ B \in \mathcal{B} \end{array} \right\},$$

where

$$\mathcal{S} \triangleq \{\mathrm{diag}(s_1 I_{k_1}, \ldots, s_\ell I_{k_\ell}) \ : \ s_i > 0\},$$

and

$$\mathcal{B} \triangleq \{\mathrm{diag}(\beta_1 I_{k_1}, \ldots, \beta_\ell I_{k_\ell}) : \beta_i \in [-\cot\theta_i, \cot\theta_i] \text{ when } \theta_i > 0\},$$

and I_k is a $k \times k$ identity matrix. (Note that the unstructured case (42.1) corresponds to $\ell = 1$.)

42.4 Available results

Concerning Statement 1, the following is shown in [2]:

- Sufficiency of (42.2) always holds. Furthermore, the following weaker "small-μ theorem" always holds: (42.2) holds if, and only if, $(I + \Delta P)^{-1} \in \mathbf{H}_\infty$ for all $\Delta \in \mathbf{B\Delta}_\theta$ and

  $$\sup_{\Delta \in \mathbf{B\Delta}_\theta} \|(I + \Delta P)^{-1}\|_\infty < \infty,$$

  where

  $$\mathbf{B\Delta}_\theta = \{\Delta \in \mathbf{H}_\infty^{n\times n} : \ \|\Delta\|_\infty \le 1,\ \Delta(\mathrm{j}\omega) \in \mathbf{\Gamma}_\theta \quad \forall \omega \in \mathbf{R}\}.$$

  Note that $\mathbf{B\Delta}_\theta$ is much larger than $\mathbf{BR\Delta}_\theta$, as it includes, among other transfer functions, all complex matrices in $\mathbf{\Gamma}_\theta$.

- In the scalar case (as well as in the diagonal uncertainty case), necessity also holds. The key is the following lemma, proved in [2]: Let $\theta \in (0, \pi/2]$, let $\hat{\omega} \in \mathbf{R} \setminus \{0\}$, and let $\gamma \in \mathbf{C}$ be such that $|\gamma| < 1$ and $|\phi(\gamma)| < \theta$. There exists $\delta \in \mathbf{RH}_\infty$ such that $\delta(\mathrm{j}\hat{\omega}) = \gamma$ and such that $\sup_{\omega \in \mathbf{R}} |\delta(\mathrm{j}\omega)| < 1$ and $\sup_{\omega \in \mathbf{R}} |\phi(\delta(\mathrm{j}\omega))| < \theta$.

It is easily verified that Statement 2 holds in the scalar case. Specifically, for $m \in \mathbf{C}$ and any $\theta \in [0, \pi/2]$, $\mu_\theta(m)$ and $\hat{\mu}_\theta(m)$ are both equal to $|m|$ when $|\phi(m)| \geq \pi - \theta$, and 0 otherwise, where $\phi(\cdot)$ is the phase taken in $(-\pi, \pi]$ and $\phi(0) = 0$.

Bibliography

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