

Chapter 7

Matrix Inequality Conditions for Canonical Factorization of Rational Transfer Function Matrices

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7.1 Description of the problem

A *canonical factorization* of a square real-rational transfer function matrix $M(s)$ is the following:

$$M(s) = M_+(s)M_-(s),$$

where $M_+(s)$, $M_+(s)^{-1}$, $M_-(-s)$, and $M_-(-s)^{-1}$ belong to \mathbf{RH}_∞ .

A related, perhaps more familiar, factorization is *spectral factorization* (see for example, [1, 2]): For a real-rational transfer function matrix $N(s)$

that satisfies $N(s) = N(-s)^T$, a spectral factorization is

$$N(s) = N_+(-s)^T N_+(s),$$

where $N_+(s)$ and $N_+(s)^{-1}$ belong to \mathbf{RH}_∞ .

Given a real-rational transfer function matrix M , it is well-known that a necessary and sufficient condition for the existence of a spectral factorization of $M(s) + M(-s)^T$ is that for some $\epsilon > 0$,

$$M(j\omega) + M(j\omega)^* \geq 2\epsilon I \text{ for all } \omega \in \mathbf{R}. \quad (7.1)$$

Given any minimal state-space realization (A, B, C, D) of M , this condition is equivalent to the existence of a real symmetric matrix P such that the following linear matrix inequality holds for some $\epsilon > 0$ (see for example, [2, 3, 4]):

$$\begin{bmatrix} A^T P + P A & P B - C^T \\ B^T P - C & 2\epsilon I - (D + D^T) \end{bmatrix} \leq 0. \quad (7.2)$$

(This is just the Kalman-Yakubovich-Popov-Anderson Lemma.)

It turns out that if condition (8.1) holds for some $\epsilon > 0$, then a *canonical* factorization exists for M . (This condition is clearly not necessary, since if M has a canonical factorization, so does $-M$, for instance.) In other words, the LMI (8.2) holding for some $\epsilon > 0$ is a sufficient condition for a transfer function $M(s)$ with a state-space realization (A, B, C, D) to have a canonical factorization.

The question is: *Given (A, B, C, D) , what, if any, are matrix inequality conditions that are both necessary and sufficient for the existence of a canonical factorization of $M(s) = C(sI - A)^{-1}B + D$?*

7.2 Motivations

Canonical factorization plays an important role in the analysis of systems with nonlinearities and/or uncertainties. A standard technique for the stability analysis of nonlinear and/or uncertain systems is the use of stability multipliers M , whose search is performed numerically [5, 6]. One of the conditions on M is that it should have a canonical factorization, and therefore simple numerical conditions that impose this constraint are important. Canonical factorization also figures in H_∞ control theory [7].

7.3 Available results

A necessary and sufficient condition for canonical factorization is given in [8] and [7, Ch. 7, Thm. 1]: M has a canonical factorization if and only if the

invariant subspace of A corresponding to its eigenvalues with positive real part and that of $A - BD^{-1}C$ corresponding to its eigenvalues with negative real part are complementary. Though this condition is very explicit, it is somewhat hard to incorporate in numerical algorithms, and hence our objective to identify conditions that are easier to check numerically.

Bibliography

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