Chapter 6

Conditions for the Existence and Uniqueness of Optimal Matrix Scalings

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6.1 Description of the problem

Given $M \in \mathbf{C}^{n\times n}$, consider the problem of finding

$$f_{\min}(M) = \inf\left\{ \left\|DMD^{-1}\right\| \;\middle|\; D \in \mathcal{D} \right\}, \tag{6.1}$$

where $\|\cdot\|$ denotes the spectral norm, and $\mathcal{D}$ is a set of "scalings" defined by

$$\mathcal{D} = \left\{ D \;\middle|\; \begin{array}{l} D \in \mathbf{C}^{n\times n},\ D = \operatorname{diag}(D_1, \ldots, D_m, d_1 I_{l_1}, \ldots, d_p I_{l_p}) \\ D_i = D_i^* \in \mathbf{C}^{k_i \times k_i},\ d_i \in \mathbf{R},\ \operatorname{Trace}(D) = 1 \end{array} \right\}. \tag{6.2}$$

Let the set of *optimal* scalings $\mathcal{D}_{\mathrm{opt}}$ be defined by

$$\mathcal{D}_{\mathrm{opt}} \triangleq \{D \mid D \in \mathcal{D}, \ \|DMD^{-1}\| = f_{\min}(M) \ \} . \tag{6.3}$$

We are interested in the following two questions:

1. When is the set $\mathcal{D}_{\mathrm{opt}}$ nonempty?
2. When is the set $\mathcal{D}_{\mathrm{opt}}$ a singleton?

An affirmative answer to the first question means that optimal scalings exist, and an affirmative answer to the second question means that there is a unique optimal scaling.

6.2 Motivations

Problem (6.1) arises in the robustness analysis of control systems with structured uncertainties. For further details, see references [1, 2]. Problem (6.1) also appears in the context of finding optimal (with various criteria for optimality) preconditioners for use in iterative algorithms; see for example, [3, 4].

6.3 Available results

For the special case when the set $\mathcal{D}$ consists of diagonal scalings (that is, when $m = 0$, and $l_1 = l_2 = \cdots = l_p = 1$ in (6.2)), partial answers to the questions are available in [5]:

1. $\mathcal{D}_{\mathrm{opt}}$ is nonempty if the matrix M is irreducible, i.e., there does not exist a permutation similarity transformation that renders M block upper-triangular. (A similar result was proved in Proposition 4 of [6] for the more general case of arbitrary l_i, with $m = 0$.)
2. For an irreducible matrix M, let D be an optimal scaling, and let the maximum singular value of DMD^{-1} be simple. Then D is the only optimal scaling if and only if there exist no pair of vectors u and v, with $\|u\| = \|v\| = 1$, satisfying

$$\begin{array}{rcl} DMD^{-1}\, v &=& \gamma\, u \\ D^{-1}M^*D\, u &=& \gamma\, v, \end{array} \tag{6.4}$$

   that belong to the same coordinate subspace, i.e., a subspace of the form $\bigcup_{i\in\mathbf{I}} \mathrm{span}\{e_i\}$, where $\mathbf{I}$ is a proper subset of the set of indices $\{1, \ldots, n\}$ and $\{e_i,\ i = 1, \ldots, n\}$ are coordinate vectors, that is, Euclidean unit vectors in $\mathbf{R}^n$.

In other words, *sufficient* conditions for the existence and uniqueness of optimal diagonal matrix scalings are given in [5]; it is also shown there that these conditions are *not* necessary.

Bibliography

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